THE UNIVERSITY OF ADELAIDE

CONVEX SETS WITH LATTICE POINT CONSTRAINTS

by

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A thesis submitted for the degree of
Doctor of Philosophy
in the Department of Pure Mathematics
The University of Adelaide

August 1996

Contents

Abstract

Every convex set in the plane gives rise to certain geometric functionals such as the area, perimeter, diameter, width, inradius and circumradius. When the convex set is constrained by lattice points (points having integer coordinates), certain inequalities occur amongst these functionals. In this thesis, we are primarily concerned with obtaining new inequalities for a planar, convex set containing exactly 0, 1 or 2 lattice points in its interior.

This thesis consists of two parts. The first part comprising Chapters 3, 4 and 5 deals with problems concerning single geometric functionals. We obtain results concerning the maximal area, circumradius and width respectively.

The second part of the thesis comprising Chapters 6 to 12 deals with a larger class of problems concerning relationships between pairs of the above-mentioned functionals for lattice constrained sets. In a number of the problems concerning 1 or 2 interior lattice points, the solution is readily obtained by reducing the problem to one concerning a set with interior containing no point of the rectangular lattice.

Chapters 1 and 2 contain basic ideas and results which are used throughout the thesis. In the concluding chapter, we comment on the scope for future research in the area. It will be seen that there remain many new and interesting problems.

Signed Statement

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Poh Wah Awyong (Miss)

August 15, 1996

Acknowledgements

I feel very fortunate to have for my friend and supervisor, Associate Professor Paul Raymond Scott. He has been a constant source of encouragement and inspiration to me. I am greatly indebted to him. Thank you also to my mother for her love and support. This thesis is dedicated to the both of them.

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List of Notation

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$\Re$	set of real numbers	2
$\mathbf{Z}$	set of integers	2
$\Re^n$	n-dimensional Euclidean space	2
$\lvert\mathbf{x}-\mathbf{y}\rvert$	Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$	2
O	origin	2
$\emptyset$	empty set	2
K^c	complement of K	2
∂K	boundary of K	2
K^o	interior of K	2
$\overline{K}$	closure of K	2
$B(\mathbf{c}, r)$	open ball centred at $\mathbf{c}$, radius r	3
AB	the line AB, or the line segment AB, or the length of the line segment AB	3
(A, B)	open line segment AB	3
$[A, B)$	$(A, B) \cup \{A\}$	3
$(A, B]$	$(A, B) \cup \{B\}$	3
$[A, B]$	$(A, B) \cup \{A, B\}$	3
λK	enlargement of K with scale factor λ	3
$\mathcal{K}^n$	set of all compact convex sets in $\Re^n$	3
$V(K) = V$	n-dimensional volume of K	3
$S(K) = S$	$(n-1)$-dimensional surface area of K	3
$A(K) = A$	area of K	3
$p(K) = p$	perimeter of K	3
$d(K) = d$	diameter of K	3
$w_{\mathbf{u}}(K)$	width of K in direction perpendicular to $\mathbf{u}$	3
$w(K) = w$	minimal width of K	3
$r(K) = r$	inradius of K	3
$R(K) = R$	circumradius of K	3

		page
$\Lambda_n(\mathbf{u}_1,\ldots,\mathbf{u}_n)$ $=\Lambda_n$	lattice generated by $\mathbf{u}_1,\ldots,\mathbf{u}_n$	4
Γ_n	the integral lattice in $\Re^n$	4
Λ	Λ_2	4
Γ	Γ_2	4
$\Lambda_R(\mathbf{u},\mathbf{v})=\Lambda_R$	lattice generated by $\mathbf{u}=(u,0)$ and $\mathbf{v}=(0,v)$	4
$\det(\Lambda_n)$	determinant of Λ_n	4
$G(K^o,\Lambda_n)$	lattice point enumerator	4
$\cong$	congruent to	10
K_l	Steiner symmetral of K with respect to l	10
$d(P,x)$	distance of point P from line x	12
Γ'	the sublattice $\{(x,y):x+y\equiv 1(\text{mod } 2)\}$	16
Γ''	the sublattice $\{(x,y):x=n,y=2m+1,m,n\in Z\}$	16
$SL1$	sublattice argument for $G(K^o,\Gamma)=1$	17
$SL2$	sublattice argument for $G(K^o,\Gamma)=2$	17
$\mathcal{E}_0$	equilateral triangle of Figure 5.1	57
$\mathcal{E}_R$	equilateral triangle of Figure 5.3	58
$\mathcal{E}_2$	equilateral triangle of Figure 5.4	59
$\mathcal{T}_0$	triangle of Figure 6.1	68
$\mathcal{T}_1$	triangle of Figure 6.2	68
$\mathcal{T}_R$	triangle of Figure 6.3	69
$\mathcal{T}_2$	triangle of Figure 6.4	70
$\mathcal{E}_1$	equilateral triangle of Figure 8.1	90
$\mathcal{S}_0$	diagonal square of Figure 10.1	98
$\mathcal{S}_1$	square of Figure 10.2	99
$\mathcal{P}_0$	parallel strip of Figure 10.3	99
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$\mathcal{H}_1$	truncated square of Figure 11.2	106
$\mathcal{H}_2$	truncated rectangle of Figure 11.3	107
$\mathcal{U}_1$	rounded square of Figure 12.1	127
$\mathcal{U}_2$	rounded hexagon of Figure 12.2	128

Chapter 1

Preliminaries

1.1 Introduction

The first ideas of convex sets date as far back as Archimedes but it was not until the end of the last century that a systematic study was made which gave rise to the subject as an independent branch of mathematics. At the turn of the century, Minkowski (1911) published his famous Convex Body Theorem which is the basis for the Geometry of Numbers. The idea is to interpret integer solutions of equations or inequalities as points with integer coordinates (lattice points). Minkowski's work provides the link between the general theory of convex sets and the geometry of numbers (concerning lattice points), giving rise to the study of convex sets with lattice point constraints.

To appreciate the sorts of questions which our research area is concerned with, we state a simple form of Minkowski's Convex Body Theorem in the plane.

Theorem 1.1 (Minkowski) *Let K be a convex set in the plane. Suppose that K is symmetric about the origin O and the interior of K contains no non-zero point whose coordinates are both integers (called a lattice point). Then its area is not greater than* 4 (Figure 1.1).

From the point of view of our research, Minkowski's Theorem is an example of a geometric extremal problem, that is, a problem concerning an inequality stated in terms of geometrical concepts. His work suggests a more general class

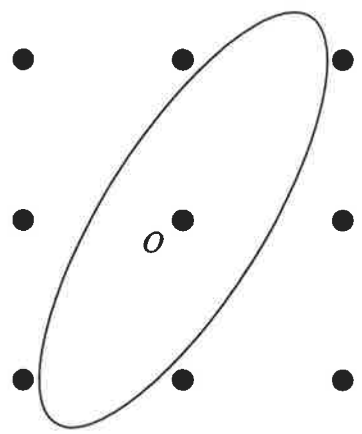


Figure 1.1: Minkowski's Convex Body Theorem

of geometric extremal problems concerning sets with lattice point constraints. Since Minkowski's Theorem, many new inequalities have been obtained for lattice constrained convex sets. An extensive bibliography is given by Croft, Falconer and Guy (1991), Erdös, Gruber and Hammer (1989), Gritzmann and Wills (1993) and Hammer (1977). This thesis is a collection of new inequalities for lattice constrained convex sets in the plane. In order to describe our problems more precisely, we now define some key terms.

1.2 Notation and Definitions

Let $\Re$ denote the set of real numbers and let $\mathbf{Z}$ denote the set of integers. Let $\Re^n$ denote the n-dimensional Euclidean space, the class of all ordered sets of n real numbers, $(x_1, x_2, \ldots, x_n)$, made into a metric space by defining the distance between any two points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ to be

$$|\mathbf{x} - \mathbf{y}| = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}.$$

We will assume simple concepts in topology.

Let O denote the origin. Let $\emptyset$ denote the empty set and let K^c denote the complement of K. The boundary, interior and closure of set K are denoted by ∂K, K^o and $\overline{K}$ respectively. The open ball with centre $\mathbf{c}$ and radius r is denoted by $B(\mathbf{c}, r)$. Let A and B be any two points in $\Re^n$. We will use AB to mean either the line AB, or the line segment AB, or the length of the line segment AB. It will be clear from the context which meaning is intended. In the case where AB denotes a line segment, we use the additional notation (A, B) to denote the line segment AB, excluding the points A and B. Similarly, we have $[A, B) = (A, B) \cup \{A\}$, $(A, B] = (A, B) \cup \{B\}$ and $[A, B] = (A, B) \cup \{A, B\}$.

The set λK, $\lambda \in \Re$ is defined to be the set $\{\lambda \mathbf{x}; \mathbf{x} \in K\}$. The scalar λ is called the *enlargement factor* of K. A set K is said to be *centrally symmetric* or simply *symmetric* about a point $\mathbf{x}$ if K is its own reflection in the point $\mathbf{x}$. Algebraically, K is centrally symmetric about $\mathbf{x}$ if for each $\mathbf{k} \in K$, $2\mathbf{x} - \mathbf{k} \in K$.

A set K is said to be *convex* if for any two points in K, the closed line segment joining the two points is contained in K. Let $\mathcal{K}^n$ denote the set of all compact (closed and bounded) convex sets in $\Re^n$.

A *support hyperplane* of K is a hyperplane which intersects K in a point on ∂K but does not intersect K^o. In $\mathcal{K}^2$, we use the term *support line.*

We now define geometric functionals on the set $\mathcal{K}^n$. We use $V(K) = V$ and $S(K) = S$ to denote the n-dimensional *volume* of K and the $(n-1)$-dimensional *surface area* of K respectively. In the case where $K \in \mathcal{K}^2$, these quantities are referred to as the *area*, denoted by $A(K) = A$, and the *perimeter*, denoted by $p(K) = p$, respectively. The *diameter* of K, denoted by $d(K) = d$, is the maximal distance between any two points of K. The *width* of K in a direction perpendicular to a given direction $\mathbf{u}$, denoted by $w_{\mathbf{u}}(K)$ is the distance between the two parallel support hyperplanes of K perpendicular to $\mathbf{u}$. The *minimal width*, $w(K) = w$, referred to simply as *width* is the minimum of $w_{\mathbf{u}}(K)$ over all $\mathbf{u}$. The *inradius* of K, $r(K) = r$, is the radius of a largest sphere contained in K. Such a sphere is called an *insphere* and the centre is called an *incentre.* The *circumradius* of K, $R(K) = R$, is the radius of the smallest sphere containing K. Such a sphere is called the *circumsphere* and its centre is called the *circumcentre.* In $\Re^2$, the insphere and the circumsphere are called the *incircle* and the *circumcircle* respectively. The circumcircle, when it exists, is unique (Yaglom and Boltyanskii 1961, p.59). The closed sets bounded by the incircle and the circumcircle are called the *indisk* and the *circumdisk* respectively.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be n linearly independent vectors in $\Re^n$. The set of points $\{\sum_{i=1}^n z_i\mathbf{u}_i; z_i \in \mathbf{Z}\}$ is called the *lattice* generated by the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, denoted by $\Lambda_n(\mathbf{u}_1, \ldots \mathbf{u}_n) = \Lambda_n$. In the case where $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is the standard basis, Λ_n is the *integral lattice* and is denoted by Γ_n. As all our problems will be posed in $\Re^2$, we will write Λ and Γ instead of Λ_2 and Γ_2 respectively. We will also use the notation $\Lambda_R(\mathbf{u},\mathbf{v}) = \Lambda_R$ to denote the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$.

Suppose that P is the set $\{\sum_{i=1}^n \alpha_i\mathbf{u}_i; 0 \leq \alpha < 1\}$. Then the set $\{P + \mathbf{z}; \mathbf{z} \in \Lambda_n\}$ is called a *cell* of Λ_n. The *determinant*, $\det(\Lambda_n)$ of Λ_n, is given by $\det(\Lambda_n) = |\det(\mathbf{u}_1, \ldots, \mathbf{u}_n)|$. Geometrically, this is the volume of a cell of Λ_n. The *lattice point enumerator*, $G(K^o, \Lambda_n)$, is defined to be the number of points of Λ_n contained in K^o. In the case where $G(K^o, \Lambda_n) = 1$ and $O \in K^o$, we say that K is Λ_n-admissible (this usage differs from the more conventional 'Λ_n is K-admissible', but is more convenient in this thesis where Λ_n is fixed and K varies). Finally, a *sublattice* is a subset of a lattice which is itself a lattice.

Other terms and definitions will be introduced in the text at the point required. We now describe the problems of the thesis.

1.3 The problems of the thesis

All our problems are posed in the Euclidean plane $\Re^2$. Unless otherwise stated, we shall henceforth assume that $K \in \mathcal{K}^2$. In this thesis we are primarily concerned with obtaining new inequalities concerning the geometric functionals A, p, d,

w, r and R for a set K in $\mathcal{K}^2$ having $G(K^o, \Gamma) = g$ where $g = 0, 1, 2$. The natural starting point for our research is to investigate problems for a set K having $G(K^o, \Gamma) = 0$. What geometric inequalities occur for such sets ? Can these results be extended to sets having $G(K^o, \Gamma) = 1$ or $G(K^o, \Gamma) = 2$? We discover that a number of such problems may be readily solved by reducing the problem to one concerning a set containing no point of a special rectangular lattice. Hence in a number of our problems, we consider the problem for a set K having $G(K^o, \Lambda_R) = 0$ and deduce the corresponding results for $G(K^o, \Gamma) = g$ where $g = 0, 1, 2$.

In Chapter 2, we describe methods and prove results which we will use throughout the thesis. Thereafter, the thesis consists essentially of two parts. Part 1 (Chapters 3, 4 and 5) deals with problems concerning single geometric parameters. The problems of Chapters 3 and 4 resulted from an attempt to prove a conjecture by Scott (1982) concerning the maximal area of a Γ-admissible set having circumcentre O. We show that the conjecture is false and we revise the conjecture. The efforts have led to the first two problems of the thesis. In Chapter 3, we obtain a result on the maximal area of a Λ-admissible set in $\mathcal{K}^2$. The result gives a classification for Λ-admissible sets. In Chapter 4, we consider Γ-admissible sets having circumcentre O. Under certain conditions, we find the maximal circumradius of such sets and we show that the extremal set is a triangle with an edge containing two lattice points (Awyong and Scott 1995). In Chapter 5, we find the maximal width of a set K with $G(K^o, \Lambda_R) = 0$ and deduce the corresponding results for $G(K^o, \Gamma) = g$ where $g = 0, 1, 2$.

Part 2 (Chapters 6-12) deals with a larger class of problems concerning relationships between pairs of the geometric parameters A, p, d, w, r and R. Clearly, from the six geometric parameters there are 15 ($= {}^6C_2$) possible combinations of two such geometric parameters.

In Chapters 6, 7, 8 and 9, we solve various problems for a set K with $G(K^o, \Lambda_R) = 0$ and deduce the corresponding results for $G(K^o, \Gamma) = g$ where $g = 0, 1, 2$. In Chapter 6 we obtain inequalities relating w and d (Awyong and Scott 1996a). This problem is motivated by a width-diameter result by Scott (1979b) for a set K having $G(K^o, \Gamma) = 0$. In Chapter 7, we generalize inequalities by Scott (1980) concerning the pairs (A, w), (p, w) and (R, w) to rectangular lattices. In Chapter 8 we find another inequality for the pair (A, w). Here we discover that the result for the case where $G(K^o, \Gamma) = 1$ may not be deduced from the case where $G(K^o, \Lambda_R) = 0$ and a conjecture is made for this case. In Chapter 9, we first obtain an inequality relating R, d and w for a set in $\mathcal{K}^2$ without lattice constraints. Using this result and the result from Chapter 5 concerning the maximal width of a set K having $G(K^o, \Lambda_R) = 0$, we derive an inequality for the pair (R, d) for a set K having $G(K^o, \Lambda_R) = 0$. As in Chapter 8, we discover that the methods do not extend to the one lattice point problem and a conjecture is made concerning the one lattice point case. We also obtain a dual inequality relating the pair (w, r).

Chapter 10 gives results relating A and r for a set K having $G(K^o, \Gamma) = 0$. We obtain the corresponding inequalities for the case $G(K^o, \Gamma) = 1$. We combine these inequalities with known inequalities in elementary geometry to deduce inequalities for the pairs (p, r) and (d, r) for a set K with $G(K^o, \Gamma) = 0$ (Awyong and Scott 1996b).

The last two problems contained in Chapters 11 and 12 concern a set K with $G(K^o, \Gamma) = 2$ and having a special symmetry condition. In Chapter 11, we establish inequalities for the pairs (A, d) and (A, R) and in Chapter 12 we find a result for the pair (A, p). We also conjecture the corresponding results for the general class of convex sets containing two interior lattice points.

Finally, in Chapter 13, we summarize the results of the thesis and make some remarks on the scope for future research in the area. It will be seen that many new and interesting problems remain in this area.

Chapter 2

Methods and results

2.1 Introduction

The theory of convex sets is one of few fields in mathematics that can be developed without the use of 'higher mathematics'. Many of the results are guided by geometric intuition and their proofs are elementary and elegant. In this chapter we describe methods and state results which we will use to prove the inequalities in this thesis. The approach we have taken in the thesis is mainly geometric and for this reason, many diagrams have been included. The numerical calculations in this thesis are carried out using Maple V, Release 2 (Copyright (C) 1981-1993 by the University of Waterloo). Graphs are plotted with Gnuplot Unix Version 3.5 (Copyright (C) 1986-1993 by Thomas Williams and Colin Kelley).

A useful technique in solving geometrical problems is to simplify the problem by applying an appropriate transformation. In §2.2 and §2.3, we describe two transformations which are of importance in the theory of convex sets as these transformations preserve the property of convexity. In §2.4, we state and prove a result called the Triangle Rotation Lemma which we will use in Chapters 5, 6 and 7. In §2.5, we describe sublattices and their role in solving problems where $G(K^o, \Gamma) = 1$ or $G(K^o, \Gamma) = 2$. Finally in §2.6, we state Blaschke's Selection Theorem, an important theorem concerning the existence of solutions to geometric extremal problems.

2.2 Affine transformation

An *affine transformation* is a linear transformation followed by a translation. We may therefore represent an affine transformation T on $\Re^2$ as follows:

$$T(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{b},$$

where $\mathbf{M}$ is a 2×2 matrix and $\mathbf{b}$ is a translation vector. We say that T is non-singular if $\det(\mathbf{M}) \neq 0$. We now state without proof some useful properties of affine transformations.

Theorem 2.1 *Let $K \in \mathcal{K}^2$ and let $T(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{b}$ be a non-singular affine transformation on $\mathcal{K}^2$. Then*

(a) $T(K) \in \mathcal{K}^2$

(b) *If K is symmetric about $\mathbf{x}$, then $T(K)$ is symmetric about $T(\mathbf{x})$*

(c) $A(T(K)) = A(K).|\det(\mathbf{M}).|$

Suppose now that T is the non-singular affine transformation $T(\mathbf{x}) = \mathbf{M}\mathbf{x}$ where $\mathbf{M} = (\mathbf{u}\ \mathbf{v})$. Then T transforms the integral lattice Γ to the lattice $\Lambda(\mathbf{u},\mathbf{v})$. By Theorem 2.1(a) and (b), T transforms a set $K \in \mathcal{K}^2$ which is symmetric about O to a set $K_* \in \mathcal{K}^2$ which is symmetric about O. Moreover, it is easy to see that if $G(K^o, \Gamma) = 0$, then $G(K^o, \Lambda) = 0$. Since $\det(\Gamma) = 1$, Theorem 2.1(c) gives

$$\frac{A(K_*)}{\det(\Lambda)} = A(K) = \frac{A(K)}{\det(\Gamma)}.$$

We say that the quantity $A/\det(\Lambda)$ is *affine-invariant*. Thus if a quantity is known to be affine-invariant, an appropriate affine transformation may be used to simplify the problem. In particular, we have the following generalization of Minkowski's Theorem for 2-dimensional lattices:

Theorem 2.2 (Generalized Minkowski) *Let K be a Λ-admissible set in $\mathcal{K}^2$ which is symmetric about the origin O. Then $A \leq 4\det(\Lambda)$.*

An affine transformation which preserves distances between points of a set is called an *isometry* (also called a rigid motion, namely reflection, rotation, translation or glide reflection). If K_i is obtained from K by an isometry, we say that K_i is *congruent* to K. We write $K_i \cong K$.

2.3 Steiner symmetrization

One of the difficulties that is encountered in working with convex sets is that convex sets are rather general figures having few special properties. Hence in our research, any method of transforming a set into another one having more special properties is very useful. *Steiner symmetrization* is one such method. In Steiner symmetrization, a convex set is transformed into a set with an axis of symmetry in the following way:

Let $K \in \mathcal{K}^2$ and let l be a given line in the plane. Let P be a point on l and let $l(P)$ be the line through P perpendicular to l. Since K is convex, $l(P)$ intersects K in a closed line segment, or in a point in ∂K or in $\emptyset$. Let $l'(P)$ be the image of $l(P) \cap K$ obtained in the following way: If $l(P) \cap K$ is a closed line segment $[X, Y]$, then $l'(P)$ is the closed line segment obtained by translating $[X, Y]$ along the line XY until P is the midpoint of $l'(P)$. If $l(P) \cap K$ is a point X in ∂K, then $l'(P)$ is the point P. If $l(P) \cap K = \emptyset$, then $l'(P) = \emptyset$. The symmetrized set, K_l, called the *Steiner symmetral with respect to* l, is now defined to be

$$K_l = \bigcup_{P \in l} l'(P).$$

We now state some useful properties of Steiner symmetrization.

Theorem 2.3 *Let $K \in \mathcal{K}^2$ and let K_l be the Steiner symmetral with respect to l. Then*

(a) $K_l \in \mathcal{K}^2$

(b) $A(K_l) = A(K)$

(c) $p(K_l) \leq p(K)$

(d) $d(K_l) \leq d(K)$

(e) $w(K)$ *may increase, decrease or be unchanged*

(f) $r(K_l) \geq r(K)$

(g) $R(K_l) \leq R(K)$.

Proof. For a proof of (a), (b), (c), (d) and (e), we refer the reader to (Eggleston 1958, p.90). We now prove (f) and (g). We first show that if $X \subseteq Y$, then $X_l \subseteq Y_l$. Let PQ be any chord of X perpendicular to l. Since $X \subseteq Y$, the line PQ intersects Y in a chord AB with $PQ \leq AB$. Now Steiner symmetrization maps chord PQ to a chord $P'Q'$ on the line PQ, with the midpoint of chord $P'Q'$ on l. Similarly, the chord AB is mapped to the chord $A'B'$ on the line PQ, with the midpoint of chord $A'B'$ on l. Since $PQ \leq AB$, the chord $P'Q'$ is a subset of the chord $A'B'$. Hence $X_l \subseteq Y_l$ (Figure 2.1).

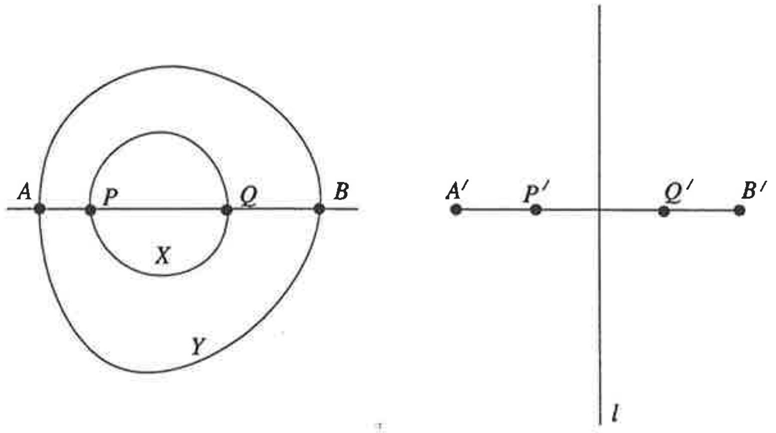


Figure 2.1: The effect of Steiner symmetrization on $r(K)$ and $R(K)$

Now let I and C be the indisk and the circumdisk respectively of K. Since $I \subseteq K \subseteq C$, it follows that $I_l \subseteq K_l \subseteq C_l$. Hence $r(K_l) \geq r(I_l)$ and $R(K_l) \leq R(C_l)$. But $I_l \cong I$ and $C_l \cong C$. Hence $r(K_l) \geq r(K)$ and $R(K_l) \leq R(K)$. □

2.4 The Triangle Rotation Lemma

In this section we state and prove an important result which we will use in Chapters 5, 6 and 7. We say that a set *circumscribes* a rectangle if all the vertices of the rectangle lie on the boundary of the set. We recall that $\Lambda_R(\mathbf{u},\mathbf{v}) = \Lambda_R$ denotes the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$. We shall denote lines with lower case letters: thus x is a line containing the lattice point X of Λ_R. Let $d(P, x)$ denote the perpendicular distance from the point P to the line x.

Lemma 2.4 (Triangle Rotation Lemma) *Let $\mathcal{C} = ABCD$, labelled anticlockwise, be a closed cell of Λ_R and let X be a lattice point on the ray from A through B. Let T be a closed triangular region (possibly an infinite triangular region) defined by lines c, d and x, and having C, D and X interior to the edges of T, with A and B not in T^o. Then either T circumscribes $\mathcal{C}$ or there is a triangle $T_* \cong \lambda T$, $\lambda > 1$, with T_* circumscribing $\mathcal{C}$.*

Proof. By a suitable rotation of the plane we may position the points A, B, C and D as shown in Figure 2.2 with $AB = u$ and $BC = v$. Let $T = \triangle PQR$ where $d.x = P$, $x.c = Q$ and $c.d = R$. We may suppose that T does not circumscribe $\mathcal{C}$. Let H_1 denote the closed half plane bounded by the line AD and containing $\mathcal{C}$. Here we distinguish the following two cases:

Case 1 : $P \in H_1$ (Figure 2.2). We first suppose that $u \leq v$. If $d(P, c) \leq v$, then by an isometry, T may be transformed to a triangle $T_1 = \triangle P_1Q_1R_1$ so that P_1 lies in $\mathcal{C}$ and Q_1R_1 lies on the line AB. Since P_1 lies in $\mathcal{C}$, T_1 may be enlarged to a triangle T_* inscribing $\mathcal{C}$. Hence $T_* \cong \lambda T$, $\lambda > 1$. If, on the other hand $d(P, c) > v$, we inscribe a rectangle R_T in T having an edge of length v and another edge of length less than u as follows: Let c' be a line parallel to c, distant v from c and intersecting edges PR and PQ in the points M' and N'

respectively. Let M and N be the feet of the perpendiculars from M' and N' respectively to the line c and let R_T be the rectangle with vertices M', N', N and M. By construction $MM' = NN' = v$. We now show that $MN = M'N' < u$.

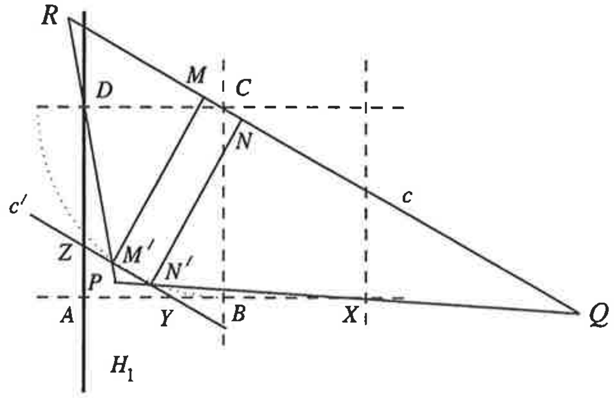


Figure 2.2: The case where $P \in H_1$

Let c' intersect the edges AD and AB in the points Z and Y respectively. Clearly $M'N' < ZY$. Letting A be the origin, we may take the coordinates of C, Z and Y to be (u, v), $(0, z)$ and $(y, 0)$ respectively. Hence

$$\text{Area of } \triangle CZY = \frac{1}{2}v.ZY = \frac{1}{2}\begin{vmatrix} u & v & 1 \\ 0 & z & 1 \\ y & 0 & 1 \end{vmatrix},$$

that is,

$$ZY = \frac{1}{v}(uz + y(v - z)).$$

Now since $0 < z < v$ and $0 < y < u$, we have

$$ZY < \frac{1}{v}(uz + u(v - z)) = \frac{1}{v}(uv) = u.$$

Hence $MN = M'N' < ZY < u$. We now rotate R_T so that $[M, N]$ lies on $[A, B]$. The same rotation transforms T to T_1, say. Clearly $G(T_1^o, \Lambda_R) = 0$ and since $M'N' < u$, at least one of C and D lies in the exterior of T_1. Hence T_1 may be enlarged to a triangle T_* inscribing the cell $\mathcal{C}$. As before $T_* \cong \lambda T$, $\lambda > 1$.

We now suppose that $u > v$. If $d(P, c) \leq u$, then by an isometry, T may be transformed to a triangle $T_1 = \triangle P_1Q_1R_1$ so that P_1 lies in $\mathcal{C}$ and Q_1R_1 lies on the line BC. As before, T_1 may be enlarged to a triangle T_* inscribing $\mathcal{C}$ and we have $T_* \cong \lambda T$, $\lambda > 1$. If now $d(P, c) > u$, we inscribe a rectangle R_T in T as described above, this time with the roles of u and v interchanged. In this case we need to show that $MN = M'N' < v$. As before, taking the coordinates of C, Z and Y to be (u, v), $(0, z)$ and $(y, 0)$ respectively, we have

$$\text{Area of } \triangle CZY = \frac{1}{2}u.ZY = \frac{1}{2}\begin{vmatrix} u & v & 1 \\ 0 & z & 1 \\ y & 0 & 1 \end{vmatrix},$$

that is,

$$ZY = \frac{1}{u}(uz + y(v - z)).$$

Since $0 < z < v$ and $0 < y < u$, we have

$$ZY < \frac{1}{u}(uz + u(v - z)) = \frac{1}{u}(uv) = v.$$

Therefore a rectangle R_T with an edge of length u and another edge of length less than v may be inscribed in T. Applying the rotation argument as before, we can find a triangle $T_* \cong \lambda T$, $\lambda > 1$, with T_* circumscribing $\mathcal{C}$.

Case 2: $P \notin H_1$. (Figures 2.3 and 2.4). Suppose first that $\angle P \leq \frac{\pi}{2}$ (Figure 2.3). Let F be the foot of the perpendicular from C to the edge PQ and let

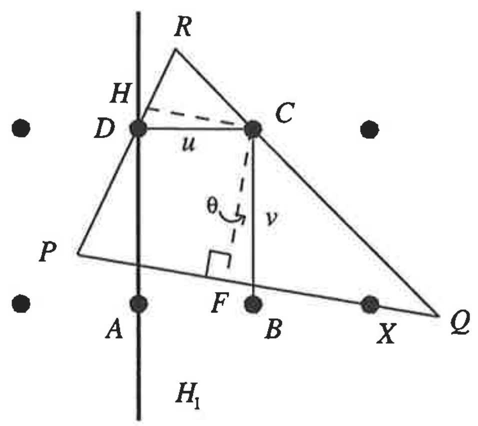


Figure 2.3: The case where $P \notin H_1$ and $\angle P \leq \frac{\pi}{2}$

H be the point on PR such that CH is parallel to PQ. Suppose that $\angle BCF = \theta$.

Then $CF \leq BC\cos\theta < BC$. Since $\angle P \leq \frac{\pi}{2}$, it follows that $\angle CHD \geq \frac{\pi}{2}$ and hence $CH < CD = u$. Hence by rotating T anticlockwise through an angle of θ about C, we obtain a triangle $T_1 = \triangle P_1Q_1R_1 \cong T$, with P_1Q_1 parallel to the line AB. We now enlarge T_1 to T_* where T_* circumscribes $\mathcal{C}$. In this case $T_* \cong \lambda T$ where $\lambda > 1$.

If now $\angle P > \frac{\pi}{2}$, we relabel the vertices of T and $\mathcal{C}$, interchanging Q with R and B with D (Figure 2.4). We define F and H as above and we let line CD intersect

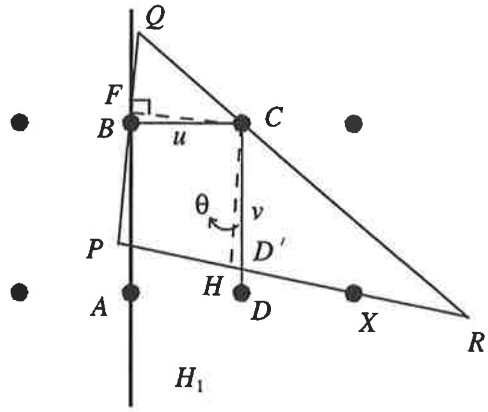


Figure 2.4: The case where $P \notin H_1$ and $\angle P > \frac{\pi}{2}$

PR in the point D'. As before, we observe that $CF < BC$. Furthermore, since $\angle P > \frac{\pi}{2}$, it follows that $\angle CHD' > \frac{\pi}{2}$. Hence $CH < CD'$. We now repeat the rotation argument above to obtain a triangle $T_1 = \triangle P_1Q_1R_1 \cong T$ (the vertices of T_1 are now labelled clockwise) with P_1Q_1 parallel to AB. We then enlarge T_1 to a triangle $T_2 = \triangle P_2Q_2R_2$ where edge P_2Q_2 contains B, edge Q_2R_2 contains C and edge P_2R_2 contains D. Hence $T_2 \cong \alpha T$, where $\alpha > 1$. Clearly, $P_2 \in H_1$ and by using the Case 1 argument, we obtain a triangle T_* circumscribing $\mathcal{C}$ with $T_* \cong \beta T_2$, $\beta > 1$. Hence $T_* \cong \lambda T$, $\lambda > 1$. □

2.5 Sublattices

By using appropriate sublattices, a number of our problems concerning one or two lattice points may be solved readily. Let

$\in \mathbf{Z}^2$ (not a lattice in the usual sense)

$$\Gamma' = \{(x,y) : x + y \equiv 1 (\text{mod } 2)\}, \quad \not\ni O$$

and let

$$\Gamma'' = \{(x,y) : x = n, y = 2m+1, m, n \in \mathbf{Z}\}.$$

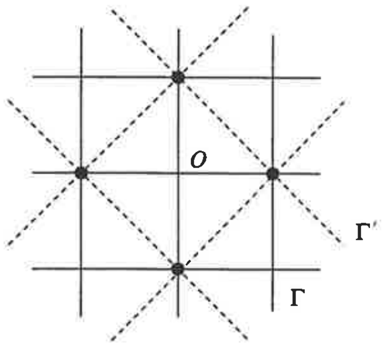


Figure 2.5: The lattice Γ'

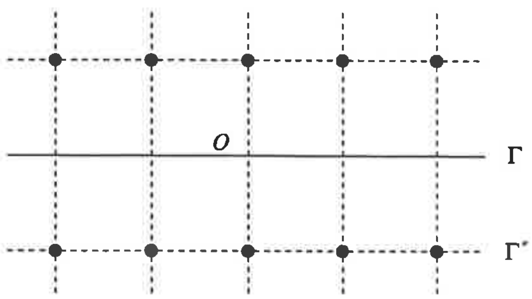


Figure 2.6: The lattice Γ''

Suppose that K is a set with $G(K^o, \Gamma) = 1$. Without losing generality, we may assume that $O \in K^o$. Then clearly $G(K^o, \Gamma') = 0$. Hence by considering the set K in the lattice Γ', the one lattice point problem in Γ is reduced to a problem concerning a set having no lattice point with respect to Γ'.

Suppose now that K is a set with $G(K^o, \Gamma) = 2$. Without losing generality, we may assume that the origin O is one of the lattice points. Let L denote the other lattice point contained in K^o and let the coordinates of L be (z_1, z_2), where without loss of generality, $z_1 \geq 0, z_2 \geq 0$. By a reflection about $y = x$ if necessary, it suffices to consider those cases for which $z_1 \geq z_2$. Since K^o contains no other lattice points, (O, L) contains no lattice points. Hence we may assume that either $z_1 = 1$ and $z_2 = 0$ or else z_1 and z_2 are relatively prime. We therefore have the following two cases:

(i) If z_1 and z_2 are both odd, we consider the sublattice Γ' and note that $O \notin \Gamma'$, $L \notin \Gamma'$ and $G(K^o, \Gamma') = 0$.

(ii) If z_1 is odd and z_2 is even, we consider the sublattice Γ''. Clearly $O \notin \Gamma''$, $L \notin \Gamma''$ and $G(K^o, \Gamma'') = 0$. In the case where z_1 is even and z_2 is odd, we consider the lattice $\Gamma''' = \{(x, y) : x = 2m + 1, y = n, m, n \in \mathbf{Z}\}$. Here, we have $G(K^o, \Gamma''') = 0$. By an appropriate transformation, this is equivalent to the case where z_1 is odd and z_2 is even.

As we will be using the results of this section frequently we use the notation $SL1$ and $SL2$ to refer to the sublattice arguments given above for the cases $G(K^o, \Gamma) = 1$ and $G(K^o, \Gamma) = 2$ respectively.

2.6 Blaschke's Selection Theorem

We now turn our attention to an important theorem concerning the existence of a solution to geometric extremal problems. The techniques used in this thesis are a combination of both constructive and non-constructive methods. In the latter case, a certain set is proved to be an extremal set by showing that any other set can be 'improved'. Such non-constructive methods are valid only if it is known that the extremal set exists. It is therefore essential to settle the question of existence before proceeding to use non-constructive methods. This is done by using Blaschke's Selection Theorem. In order to understand how the theorem answers the existence question, we first recall the following theorem from elementary topology.

Theorem 2.5 *Let f be a continuous real-valued function on a compact subset S of a metric space X. Then f attains its maximum and minimum on the set S.*

In order to use Theorem 2.5 to establish the existence of an extremal set, we first need to make $\mathcal{K}^2$ a metric space. We do this by defining the following distance function, D (also called the Hausdorff metric) on $\mathcal{K}$:

$$D(X,Y) = \inf\{\delta : X \subseteq Y_\delta, Y \subseteq X_\delta\}.$$

where $A_\delta = \bigcup_{a\in A} \overline{B}(a,\delta)$. Now let $S \subseteq \mathcal{K}^2$ and let f be a continuous real-valued function on S. Then by Theorem 2.5, extremal sets are attained in S if S can be shown to be compact (a set S is compact if every infinite sequence in S has a subsequence which converges to a point in S). Blaschke's Selection Theorem gives a criterion for compactness in the metric space $\mathcal{K}^2$.

Theorem 2.6 (Blaschke's Selection Theorem) *Let $\{K_i\}$ be an infinite sequence of sets in $\mathcal{K}^2$ lying in a disk in $\Re^2$. Then there is a subsequence which converges to a set in $\mathcal{K}^2$.*

Hence if the members of S may be shown to be contained in a disk in $\Re^2$, then by Blaschke's Selection Theorem, every infinite sequence in S has a subsequence which converges to a point in $\mathcal{K}^2$. If, in addition, it can be shown that the limit set is in fact in S, then S is a compact subset in $\mathcal{K}^2$ and by Theorem 2.5, there are sets S_M and S_m in S such that for all $K \in S$,

$$f(S_m) \leq f(K) \leq f(S_M).$$

We call S_m a *minimal* set and S_M a *maximal* set.

Chapter 3

A classification for planar convex sets containing one lattice point

3.1 Introduction

Let K be a Λ_n-admissible set in $\mathcal{K}^n$ with volume $V(K) = V$ (in $\mathcal{K}^2$, the volume is referred to as the area, denoted by $A(K) = A$). Minkowski's Convex Body Theorem as stated in Theorem 1.1 gives a result for the maximal area of a Γ-admissible set in $\mathcal{K}^2$ which is symmetric about O. A large number of results concerning the maximal volume of asymmetric sets in $\mathcal{K}^n$ containing no non-zero lattice points appear in the literature. We mention some of these in §3.5. In particular, Ehrhart (1964) conjectures the following result:

Conjecture 3.1 (Ehrhart) *Let K be a Λ_n-admissible set in $\mathcal{K}^n$ and let O be the centre of gravity of K. Then*

$$V \leq \frac{(n+1)^n}{n!} \det(\Lambda_n).$$

Equality holds when and only when K is a simplex.

Ehrhart (1955a, 1955b) proves the above result for the cases where $n = 2$ and for the class of solids of revolution for $n = 3$ respectively. In the case where $n = 2$, equality is attained when and only when K is the triangle having vertices $(-2,-1)$, $(1,-1)$ and $(1,2)$ (Figure 3.1). We call this triangle Ehrhart's triangle and denote it by E.

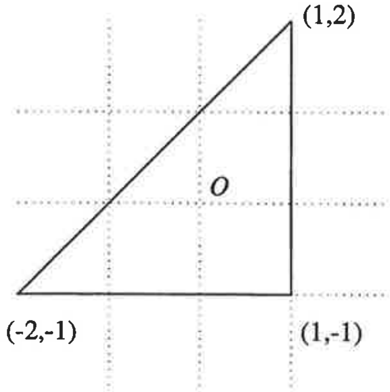


Figure 3.1: Ehrhart's triangle, E

In this chapter, we establish an easy test to determine the maximal area of various classes of sets in $\mathcal{K}^2$ containing one interior lattice point. This test both extends and generalizes Minkowski's theorem. It also brings into clearer focus the role of Ehrhart's triangle. The test results in a classification of such sets into three basic types:

Type I: Sets for which $A \leq 4\det(\Lambda)$. An example for $\Lambda = \Gamma$ is the square S having vertices $(\pm 1, \pm 1)$.

Type II: Sets for which $A \leq 4\frac{1}{2}\det(\Lambda)$. An example for $\Lambda = \Gamma$ is Ehrhart's triangle E.

Type III: Sets for which A is unbounded. An example for $\Lambda = \Gamma$ is the set bounded by $y = -1$, and the two near-collinear rays with common endpoint $(0, \varepsilon)$ (where ε is small and positive), and passing through $(1, 0)$ and $(-1, 0)$ respectively.

Let Λ be generated by two vectors $\mathbf{u}$ and $\mathbf{v}$, and let P be the parallelogram having vertices ($\pm\mathbf{u}$, $\pm\mathbf{v}$). Our test will involve the intercepts which set K makes with the edges of P. Since the ratio $A/\det(\Lambda)$ is invariant under affine transformation (see §2.2), it will be sufficient for us to state and prove our result for the case where $\Lambda = \Gamma$ and P is the square S.

So as above, let S be the closed square with vertices $(\pm 1, \pm 1)$. We label the eight lattice points $(1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1), (1,0)$ on S, $L_1, L_2, \ldots, L_8$ respectively. In §3.3, the lattice point $(-2,-1)$ is of special interest; we denote the point by L_0. The half edges, $L_1L_2, L_2L_3, \ldots, L_8L_1$, of S are labelled h_1, h_2, ... , h_8 respectively. We shall say that one set of half edges is *equivalent* to another set of half edges if the second can be obtained from the first under an isometry.

If K lies in S, then $A \leq 4$ and K will be a set of Type I. We may therefore suppose that K extends beyond ∂S. We say that K *crosses* an half edge h_i of S if h_i contains a point in K^o. Since by convexity K cannot cross two half edges comprising an edge of S, K may cross at most four of the eight half edges of S. For a given set K, we call the set of half edges of S which are crossed by K the *intercept set* of K. We shall establish the following result.

Theorem 3.2 *Let K be a set in $\mathcal{K}^2$. If the intercept set of K is $\emptyset$ or equivalent to $\{h_1, h_3, h_5, h_7\}$, then $A \leq 4$. If the intercept set is equivalent to $\{h_1, h_4, h_7\}$, $\{h_1, h_4, h_5, h_7\}$, $\{h_1, h_4, h_5, h_8\}$ or $\{h_1, h_4, h_6, h_7\}$, then $A < 4\frac{1}{2}$. Otherwise A is unbounded.*

We have included some helpful diagrams in Appendix A. These diagrams will be referred to in the subsequent sections.

3.2 The unbounded cases

Let $l_i(\theta_i)$ be a line containing the lattice point L_i and making a directed angle of θ_i, $-\frac{\pi}{2} < \theta_i \leq \frac{\pi}{2}$, with the positive x-axis.

In the one-intercept case, we may take the intercept set to be $\{h_1\}$, as all intercept sets $\{h_i\}$ are equivalent. We construct a convex polygonal set K bounded by the lines $x = 1$, $y = -1$ and the line $l_2(\theta_2)$ with $\theta_2 = \frac{\pi}{2} - \varepsilon$, where ε is small and positive. Clearly, K is Γ-admissible and $A \to \infty$ as $\varepsilon \to 0$.

If K crosses exactly two half edges of S, using rotation about the origin and reflection in a line through the origin, it suffices to consider the cases where the intercept set of K is one of

(a) $\{h_1, h_3\}$ (b) $\{h_1, h_4\}$ (c) $\{h_1, h_5\}$ (d) $\{h_1, h_6\}$ (e) $\{h_1, h_8\}$.

In case (a), we construct a convex polygonal set K bounded by the lines $x = 1$, $l_2(\theta_2)$ with $\theta_2 = \varepsilon$ and $l_4(\theta_4)$ with $\theta_4 = -\varepsilon$ (where ε is small and positive). Again, K is Γ-admissible and $A \to \infty$ as $\varepsilon \to 0$. In a similar way, sets with arbitrarily large areas may be constructed for the remaining two-intercept cases (Appendix A, Figure A.1).

If K crosses exactly three half edges, using rotation about the origin and reflection in a line through the origin, it suffices to consider the cases where the intercept set of K is one of

(a) $\{h_1, h_3, h_5\}$ (b) $\{h_1, h_3, h_8\}$ (c) $\{h_1, h_4, h_7\}$ (d) $\{h_1, h_4, h_8\}$.

In case (a), we construct the convex set K bounded by lines $l_1(\theta_1)$, $l_2(\theta_2)$ and $l_4(\theta_4)$, with $\theta_1 = \frac{\pi}{4}+\varepsilon$, $\theta_2 = \frac{\pi}{4}-\varepsilon$ and $\theta_4 = -\frac{\pi}{2}+\varepsilon'$ (where ε and ε' are arbitrarily small and positive). This set K is Γ-admissible and $A \to \infty$ as $\varepsilon \to 0$ and $\varepsilon' \to 0$. In a similar way, sets with arbitrarily large areas for cases (b) and (d) may be constructed (Appendix A, Figure A.2). In §3.3, we deal with the outstanding $\{h_1, h_4, h_7\}$ case above. In §3.4, we consider the cases where K intercepts four half edges.

3.3 The $\{h_1, h_4, h_7\}$ case

Let I denote the lattice point index set $\{0, 1, 2, \ldots, 8\}$. Since K is Γ-admissible, $L_i \notin K^o$ for $i \in I$. Therefore, since K is convex, for a suitable choice of θ_i for each i, $l_i(\theta_i)$ does not intersect K^o. Let π_i denote the closed half plane containing K and having boundary $l_i(\theta_i)$. We will show that for each K having intercept set $\{h_1, h_4, h_7\}$, a set K^* may be constructed with the following properties:

(a) $K \subseteq K^*$

(b) K^* is Γ-admissible

(c) K^* has the same intercept set as K

(d) $K^* = \bigcap_{j \in J} \pi_j$, $J \subseteq I$.

We first observe that since K is convex and crosses the half edges h_1 and h_7, $K \subseteq \pi_8$ with $\theta_8 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$. We shall take π_8 with the associated range of values for θ_8 as a defining half plane for K^*. Further defining half planes π_i with corresponding ranges for θ_i will be selected in the same way. We now carefully enumerate the possibilities for K^* by considering the different ways in which K^o intersects the line $y = x + 1$. Since K is Γ-admissible and K has intercept set $\{h_1, h_4, h_7\}$, K^o intersects the line $y = x + 1$ in one of the following four ways (Appendix A, Figure A.3):

1. in $\emptyset$ or between the points $(1, 2)$ and L_2. Therefore $K \subseteq \pi_2$ with $\theta_2 \in [\frac{\pi}{4}, \arctan 2)$. This gives rise to two subcases. The set K^o intersects $y = -1$ in one of the following ways:

   (i) in $\emptyset$ or between the points L_0 and L_5. In this case $K \subseteq \pi_5$ with $\theta_5 \in [0, \arctan \frac{1}{2})$. Here we set $K^* = \pi_2 \cap \pi_5 \cap \pi_8$.

   (ii) between the points L_7 and $(2, -1)$. In this case, $K \subseteq \pi_7$ with $\theta_7 \in (-\arctan \frac{1}{2}, 0)$. Here we set $K^* = \pi_2 \cap \pi_7 \cap \pi_8$;

2. between the points L_2 and L_4. Therefore $K \subseteq \pi_2 \cap \pi_4$ with $\theta_2 \in (0, \frac{\pi}{4})$ and $\theta_4 \in (\frac{\pi}{4}, \frac{\pi}{2})$. The subcases here are as for (1) above. This gives rise to two possibilities for K^*, namely

   (i) $\pi_2 \cap \pi_4 \cap \pi_5 \cap \pi_8$ with $\theta_5 \in [0, \arctan \frac{1}{2})$, or

   (ii) $\pi_2 \cap \pi_4 \cap \pi_7 \cap \pi_8$ with $\theta_7 \in (-\arctan \frac{1}{2}, 0)$;

3. between the points L_4 and L_0. In this case $K \subseteq \pi_4$ with $\theta_4 \in (\arctan\frac{1}{2}, \frac{\pi}{4})$. Here we have three subcases. The set K^o intersects $y = -1$ in one of the following ways:

   (i) in $\emptyset$ or between the points $(-3, -1)$ and L_0; in this case $K \subseteq \pi_0$ with $\theta_0 \in [0, \arctan\frac{1}{3})$. We set $K^* = \pi_4 \cap \pi_8 \cap \pi_0$.

   (ii) between the points L_0 and L_5; in this case $K \subseteq \pi_0 \cap \pi_5$ with $\theta_5 \in (0, \arctan\frac{1}{2})$ and $\theta_0 \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (-\frac{\pi}{2}, 0)$. We set $K^* = \pi_4 \cap \pi_5 \cap \pi_8 \cap \pi_0$.

   (iii) between the points L_7 and $(2, -1)$; in this case $K \subseteq \pi_7$ with $\theta_7 \in (-\arctan\frac{1}{2}, 0)$. We set $K^* = \pi_4 \cap \pi_7 \cap \pi_8$;

4. between the points L_0 and $(-3, -2)$. Here $K \subseteq \pi_0$ with $\theta_0 \in (\arctan\frac{2}{3}, \frac{\pi}{4})$. In this case, K^o intersects $y = -1$ between the points L_0 and L_5. Hence $K \subseteq \pi_5$ with $\theta_5 \in (0, \arctan\frac{1}{2})$. We set $K^* = \pi_5 \cap \pi_8 \cap \pi_0$.

It may be easily verified in each case that K^* satisfies properties (a), (b), (c) and (d) listed earlier. Since $K \subseteq K^*$, $A(K) \leq A(K^*)$. It is therefore sufficient to prove Theorem 3.2 for K^*. Henceforth we shall assume that $K = K^*$.

In each of these cases, K extends beyond Ehrhart's triangle E in a set $K \backslash (K \cap E)$. We consider the decomposition of $K \backslash (K \cap E)$ into a finite number of triangles 'cut off' from K by the lines $y = -1$, $x = 1$ and $y = x + 1$ bounding E. Let $\triangle$ denote the set of such triangles. Since the intercept set of K is $\{h_1, h_4, h_7\}$, each member of $\triangle$ lies in one of the closed triangular regions $J_1, J_2, \ldots, J_8$ shown in Figure 3.2. We denote by $\triangle_i$ the member of $\triangle$ lying in J_i. We will associate with each $\triangle_i \in \triangle$ a triangle $T_i \subset E \backslash (K \cap E)$ having $A(\triangle_i) \leq A(T_i)$ and $T_i \cap T_j = \emptyset$ if $i \neq j$. This will show that $A(K) \leq A(E)$. The triangles T_i and $\triangle_i$ will have a common vertex, and vertically opposite angles at that vertex. Thus given the vertex, the lines on which two edges of T_i lie will be automatically determined and T_i will be completely specified by its third edge.

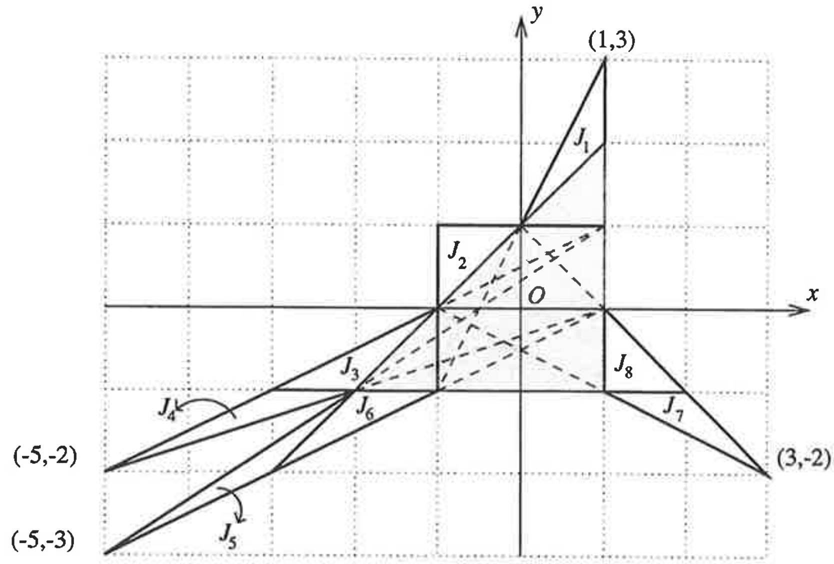


Figure 3.2: The regions $J_1, J_2, \ldots, J_8$ for the $\{h_1, h_4, h_7\}$ case

For each $\Delta_i \in \Delta$, we choose the common vertex of Δ_i and T_i and the line on which the third edge of T_i lies as given in Table 3.1 (see also Appendix A, Figure A.3).

Triangles	Common vertex	Third edge of T_i lies on
Δ_1, T_1	L_2	$x = -1$
Δ_2, T_2	L_2	$x = 1$
Δ_3, T_3	L_4	$y = 1$
Δ_4, T_4	L_0	$x = 1$
Δ_5, T_5	L_0	$y = 1$
Δ_6, T_6	L_5	$x = 1$
Δ_7, T_7	L_7	$x = -1$
Δ_8, T_8	L_8	$y = 1.$

Table 3.1: Triangles Δ_i and T_i for the $\{h_1, h_4, h_7\}$ case

Hence, if for example, $K = \pi_2 \cap \pi_5 \cap \pi_8$ with $\theta_2 \in [\frac{\pi}{4}, \arctan 2)$, $\theta_5 \in [0, \arctan \frac{1}{2})$ and $\theta_8 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$, then K intercepts J_1, J_6 and J_8 and $\Delta = \{\Delta_i, i = 1, 6, 8\}$. The triangles T_i, $i = 1, 6, 8$ are chosen as in Table 3.1. It may be easily checked here that for each i, $A(T_i) \geq A(\Delta_i)$ and $T_i \cap T_j = \emptyset$ if $i \neq j$. We repeat the process for each of the remaining cases for K listed above, again finding that for each (relevant) i, $A(\triangle_i) \leq A(T_i)$ and $T_i \cap T_j = \emptyset$ if $i \neq j$. Since at least one of the triangle area inequalities is strict, it follows that $A(K) < A(E) = 4\frac{1}{2}$ for all K having intercept set $\{h_1, h_4, h_7\}$.

We now show that $A(E) = 4\frac{1}{2}$ is in fact the least upper bound for $A(K)$. Consider the infinite sequence of triangles $\{K_r\}$ where $K_r = \pi_2 \cap \pi_5 \cap \pi_8$ with $\theta_2 = \frac{\pi}{4}$, $\theta_5 = 0$ and $\theta_8 = -\arctan 2^r$. Then as $r \to \infty$, $K_r \to E$. Since $A(K_r)$ is a continuous function of r, $A(K_r) \to A(E) = 4\frac{1}{2}$. Hence the least upper bound for $A(K)$ is $4\frac{1}{2}$.

3.4 The four-intercept case

If K crosses exactly four half edges, using rotation about the origin and reflection in a line through the origin, it suffices to consider the cases where the intercept set of K is one of

(a) $\{h_1, h_3, h_5, h_7\}$ (b) $\{h_1, h_4, h_5, h_7\}$ (c) $\{h_1, h_4, h_6, h_7\}$

(d) $\{h_1, h_4, h_5, h_8\}$.

We will employ a similar method to that given in §3.3 to prove the results for the four-intercept case. We will also use the notation employed in §3.3.

Suppose first that K has intercept set $\{h_1, h_3, h_5, h_7\}$. In this case, we observe that $K \subseteq \pi_2 \cap \pi_4 \cap \pi_6 \cap \pi_8$ with $\theta_2, \theta_6 \in (0, \frac{\pi}{4})$, $\theta_4, \theta_8 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$. Hence we set $K^* = \pi_2 \cap \pi_4 \cap \pi_6 \cap \pi_8$. Now replacing K by K^* and recalling that S is the square bounded by the lines $x = \pm 1$ and $y = \pm 1$, we observe that K extends beyond S in a set $K \backslash (K \cap S)$. We will show that $A(S) = 4$ is the least upper bound for $A(K)$.

We consider the decomposition of $K \backslash (K \cap S)$ into four triangles 'cut off' from K by the lines bounding S. We observe that each member of $\triangle$ lies in one of the closed triangular regions J_1, J_2, J_3, J_4 shown in Figure 3.3. Hence

$\Delta = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. We now choose the common vertex of Δ_i and T_i and the line on which the third edge of T_i lies as given in Table 3.2.

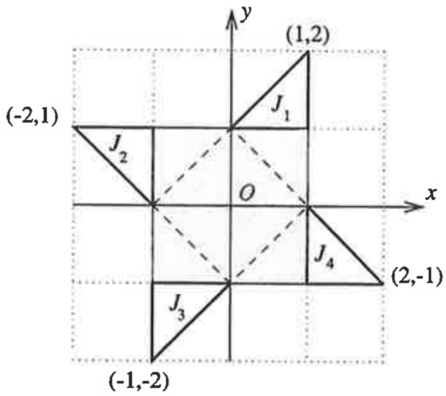


Figure 3.3: The regions $J_1, J_2, \ldots J_4$ for the $\{h_1, h_3, h_5, h_7\}$ case

Triangles	Common vertex	Third edge of T_i lies on
Δ_1, T_1	L_2	$x = -1$
Δ_2, T_2	L_4	$y = -1$
Δ_3, T_3	L_6	$x = 1$
Δ_4, T_4	L_8	$y = 1$

Table 3.2: Triangles Δ_i and T_i for the $\{h_1, h_3, h_5, h_7\}$ case

It is a simple matter to show that for each i, $A(\Delta_i) < A(T_i)$ and $T_i \cap T_j = \emptyset$ for $i \neq j$. Hence $A(K) < A(S) = 4$ for all K having intercept set $\{h_1, h_3, h_5, h_7\}$.

We now show that $A(S) = 4$ is in fact the least upper bound for $A(K)$. We consider the infinite sequence of squares $\{K_r\}$ where $K_r = \pi_2 \cap \pi_4 \cap \pi_6 \cap \pi_8$ with $\theta_2 = \theta_6 = \arctan 2^{-r}$ and $\theta_4 = \theta_8 = -\arctan 2^r$. As $r \to \infty$, $K_r \to S$ and $A(K_r) \to A(S) = 4$. Hence the least upper bound for $A(K)$ is 4.

Now suppose that K has intercept set $\{h_1, h_4, h_5, h_7\}$. Here we note that $K \subseteq \pi_5 \cap \pi_6 \cap \pi_8$ with $\theta_5 \in (-\frac{\pi}{2}, 0)$, $\theta_6 \in (0, \frac{\pi}{4})$ and $\theta_8 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$. We next observe that K^o may intersect the line $y = x + 1$ in one of the following ways (Appendix A, Figure A.4):

1. in $\emptyset$ or between the points $(1,2)$ and L_2. Hence $K \subseteq \pi_2$ with $\theta_2 \in [\frac{\pi}{4}, \arctan 2)$. We set $K^* = \pi_2 \cap \pi_5 \cap \pi_6 \cap \pi_8$;
2. between the points L_2 and L_4. Hence $K \subseteq \pi_2 \cap \pi_4$ with $\theta_2 \in (0, \frac{\pi}{4})$ and $\theta_4 \in (\frac{\pi}{4}, \frac{\pi}{2})$. We set $K^* = \pi_2 \cap \pi_4 \cap \pi_5 \cap \pi_6 \cap \pi_8$;
3. between the points L_4 and L_0. Hence $K \subseteq \pi_4$ with $\theta_4 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$. We set $K^* = \pi_4 \cap \pi_5 \cap \pi_6 \cap \pi_8$.

We show that an upper bound for $A(K)$ is $A(E) = 4\frac{1}{2}$ and that this is in fact the least upper bound. We consider the decomposition of $K \backslash (K \cap E)$ into a finite number of triangles 'cut off' from K by the lines bounding E. Taking note of the intercept set of K, we observe that each member of $\triangle$ lies in one of the closed triangular regions J_1, J_2, J_3, J_4, J_5 shown in Figure 3.4. We now construct the set $\triangle$ for each of the edge sets above. For each $\triangle_i \in \triangle$, we choose the common vertex of $\triangle_i$ and T_i and the line on which the third edge of T_i lies as given in Table 3.3 (see also Appendix A, Figure A.4).

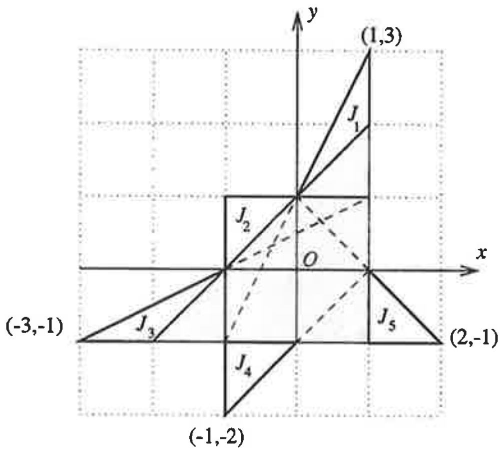


Figure 3.4: The regions $J_1, J_2, \ldots, J_5$ for the $\{h_1, h_4, h_5, h_7\}$ case

It may be easily checked that for each $\triangle_i \in \triangle$, $A(\triangle_i) \leq A(T_i)$ (with strict inequality for at least one i), and $T_i \cap T_j = \emptyset$ for $i \neq j$. Hence $A(K) < A(E) = 4\frac{1}{2}$ for all K having intercept set $\{h_1, h_4, h_5, h_7\}$.

Triangles	Common vertex	Third edge of T_i lies on
$\triangle_1, T_1$	L_2	$x = -1$
$\triangle_2, T_2$	L_2	$x = 1$
$\triangle_3, T_3$	L_4	$y = 1$
$\triangle_4, T_4$	L_6	$x = 1$
$\triangle_5, T_5$	L_8	$y = 1$

Table 3.3: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_7\}$ case

To show that $4\frac{1}{2}$ is the least upper bound for $A(K)$, we consider the infinite sequence $\{K_r\}$ where $K_r = \pi_2 \cap \pi_5 \cap \pi_6 \cap \pi_8$ with $\theta_2 = \frac{\pi}{4}$, $\theta_5 = -\arctan 2^{-r}$, $\theta_6 = \arctan 2^{-r}$ and $\theta_8 = -\arctan 2^r$. As $r \to \infty$, $K_r \to E$ and $A(K_r) \to A(E)$. Hence $A(E) = 4\frac{1}{2}$ is the least upper bound for $A(K)$.

If K has intercept set $\{h_1, h_4, h_6, h_7\}$, we have $K \subseteq \pi_6 \cap \pi_7 \cap \pi_8$ with $\theta_6 \in (-\frac{\pi}{4}, 0)$, $\theta_7 \in (0, \frac{\pi}{2})$ and $\theta_8 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$. Noting that K^o intercepts $y = x + 1$ as in the $\{h_1, h_4, h_5, h_7\}$ case, we have the following possibilities for K^*, namely

1. $\pi_2 \cap \pi_6 \cap \pi_7 \cap \pi_8$ with $\theta_2 \in [\frac{\pi}{4}, \arctan 2)$,

2. $\pi_2 \cap \pi_4 \cap \pi_6 \cap \pi_7 \cap \pi_8$ with $\theta_2 \in (0, \frac{\pi}{4})$ and $\theta_4 \in (\frac{\pi}{4}, \frac{\pi}{2})$,

3. $\pi_4 \cap \pi_6 \cap \pi_7 \cap \pi_8$ with $\theta_4 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$.

We observe that the figure in this case is the same as Figure 3.4 with J_4 reflected in the y-axis. The choice of common vertices for $\triangle_i$ and T_i and the line on which the third edge of T_i lies is therefore the same as for the $\{h_1, h_4, h_5, h_7\}$ case for $i = 1, 2, 3, 5$. For $\triangle_4$ and T_4, the common vertex is L_6 and the third edge of T_4 is chosen to lie on $x = -1$. It may then be argued in the same way as for the $\{h_1, h_4, h_5, h_7\}$ case that $A(E) = 4\frac{1}{2}$ is an upper bound for $A(K)$.

To show that $A(E) = 4\frac{1}{2}$ is the least upper bound for $A(K)$, we consider the infinite sequence $\{K_r\}$ where $K_r = \pi_2 \cap \pi_6 \cap \pi_7 \cap \pi_8$ with $\theta_2 = \frac{\pi}{4}$, $\theta_6 = -\arctan 2^{-r}$,

$\theta_7 = \arctan 2^{-r}$ and $\theta_8 = -\arctan 2^r$. As $r \to \infty$, $K_r \to E$. Therefore $A(E) = 4\frac{1}{2}$ is the least upper bound for $A(K)$.

Finally in the $\{h_1, h_4, h_5, h_8\}$ case, we note that $K \subseteq \pi_1 \cap \pi_5$ with $\theta_1 \in (-\frac{\pi}{2}, 0)$ and $\theta_5 \in (-\frac{\pi}{2}, 0)$. We first observe that K^o intercepts the line $y = x+1$ as in cases (b) and (c) above. Hence we have the following cases (Appendix A, Figure A.5):

1. $K \subseteq \pi_2$ with $\theta_2 \in [\frac{\pi}{4}, \arctan 2)$. This gives rise to three subcases. The set K^o may intersect the line $y = x - 1$ in one of the following three ways:

   (i) in $\emptyset$ or between the points $(-1,-2)$ and L_6. In this case $K \subseteq \pi_6$ with $\theta_6 \in [\frac{\pi}{4}, \arctan 2)$. We set $K^* = \pi_1 \cap \pi_2 \cap \pi_5 \cap \pi_6$;

   (ii) between the points L_6 and L_8. In this case $K \subseteq \pi_6 \cap \pi_8$ with $\theta_6 \in (0, \frac{\pi}{4})$ and $\theta_8 \in (\frac{\pi}{4}, \frac{\pi}{2})$. We set $K^* = \pi_1 \cap \pi_2 \cap \pi_5 \cap \pi_6 \cap \pi_8$;

   (iii) between the points L_8 and $(2, 1)$. In this case $K \subseteq \pi_8$ with $\theta_8 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$. We set $K^* = \pi_1 \cap \pi_2 \cap \pi_5 \cap \pi_8$.

2. $K \subseteq \pi_2 \cap \pi_4$ with $\theta_2 \in (0, \frac{\pi}{4})$ and $\theta_4 \in (\frac{\pi}{4}, \frac{\pi}{2})$. The same subcases as in (1) above arise, giving the following possibilities for K^*.

   (i) $\pi_1 \cap \pi_2 \cap \pi_4 \cap \pi_5 \cap \pi_6$ with $\theta_6 \in [\frac{\pi}{4}, \arctan 2)$;

   (ii) $\pi_1 \cap \pi_2 \cap \pi_4 \cap \pi_5 \cap \pi_6 \cap \pi_8$ with $\theta_6 \in (0, \frac{\pi}{4})$ and $\theta_8 \in (\frac{\pi}{4}, \frac{\pi}{2})$;

   (iii) $\pi_1 \cap \pi_2 \cap \pi_4 \cap \pi_5 \cap \pi_8$ with $\theta_8 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$.

3. $K \subseteq \pi_4$ with $\theta_4 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$. The same subcases as (1) arise, giving the following three possibilities for K^*.

   (i) $\pi_1 \cap \pi_4 \cap \pi_5 \cap \pi_6$ with $\theta_6 \in [\frac{\pi}{4}, \arctan 2)$;

   (ii) $\pi_1 \cap \pi_4 \cap \pi_5 \cap \pi_6 \cap \pi_8$ with $\theta_6 \in (0, \frac{\pi}{4})$ and $\theta_8 \in (\frac{\pi}{4}, \frac{\pi}{2})$;

   (iii) $\pi_1 \cap \pi_4 \cap \pi_5 \cap \pi_8$ with $\theta_8 \in (\arctan \frac{1}{2}, \frac{\pi}{4})$.

Noting that the lattice points L_1, L_2, L_4, L_5, L_6, L_8 and the half edges h_1, h_4, h_5, h_8 are symmetrically placed about the origin, we observe that case 3(iii) may be incorporated into case 1(i); case 1(iii) may be incorporated into case 3(i) and cases 1(ii), 2(iii) and 3(ii) may be incorporated into case 2(i). Therefore we need only consider the remaining four cases for K^*, namely cases 1(i), 2(i), 2(ii) and 3(i).

We now consider the decomposition of $K \backslash (K \cap E)$ into a finite number of triangles cut off from K by the lines bounding E. Given the intercept set of K in this case, we note that each member of $\triangle$ lies in one of the closed triangular regions $J_1, J_2, \ldots, J_5$ shown in Figure 3.5. We choose the common vertex of $\triangle_i$ and T_i and the line on which the third edge of T_i lies as given in Table 3.4 (see also Appendix A, Figure A.5).

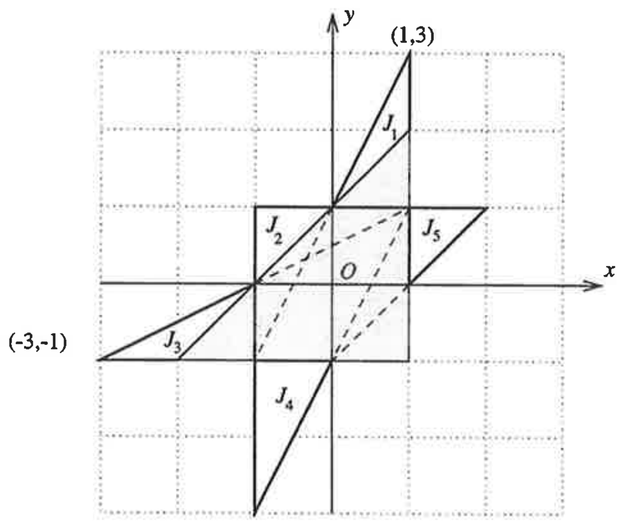


Figure 3.5: The regions $J_1, J_2, \ldots, J_5$ for the $\{h_1, h_4, h_5, h_8\}$ case

It may be easily checked that by constructing the set $\triangle$ for each of the edge sets for K listed above, $A(\triangle_i) \leq A(T_i)$ for each i (with strict inequality for at least one i), and $T_i \cap T_j = \emptyset$ for $i \neq j$. Hence $A(K) < A(E) = 4\frac{1}{2}$ for all K having intercept set $\{h_1, h_4, h_5, h_8\}$.

Triangles	Common vertex	Third edge of T_i lies on
$\triangle_1, T_1$	L_2	$x = -1$
$\triangle_2, T_2$	L_4	$y = -1$
$\triangle_3, T_3$	L_4	$y = 1$
$\triangle_4, T_4$	L_6	$x = 1$
$\triangle_5, T_5$	L_1	$y = x + 1$

Table 3.4: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_8\}$ case

To show that $A(E) = 4\frac{1}{2}$ is the least upper bound for $A(K)$, we consider an infinite sequence $\{K_r\}$ where $K_r = \pi_1 \cap \pi_2 \cap \pi_5 \cap \pi_6 \cap \pi_8$ with $\theta_1 = -\arctan 2^r$, $\theta_2 = \frac{\pi}{4}$, $\theta_5 = -\arctan 2^{-r}$, $\theta_6 = \arctan 2^{-r}$ and $\theta_8 = \arctan 2^r$. As $r \to \infty$, $K_r \to E$ and $A(K_r) \to A(E) = 4\frac{1}{2}$. Hence $A(E) = 4\frac{1}{2}$ is the least upper bound for $A(K)$.

This completes the proof, and Theorem 3.2 is established.

3.5 Related results and conjectures

Many interesting problems concerning the maximal volume of a set K in $\mathcal{K}^n$ arise when conditions are imposed on the set K to keep the volume bounded. We have already mentioned the results of Minkowski (1911) and Ehrhart (1955a, 1955b) as examples. We state here other significant results concerning the maximal volume of a Λ_n-admissible set. The results given here are by no means exhaustive and we refer the reader to the surveys by Erdös, Gruber and Hammer (1989), Gritzmann and Wills (1993), Hammer (1977) and Scott (1988) for a more complete coverage.

Sawyer (1954) defines a functional $\lambda(K)$ for a Λ_n-admissible set K in $\mathcal{K}^n$ as follows: Let POP' be an arbitrary chord of K. Then $\lambda(K) = \sup PO/OP'$. It is clear that $\lambda(K) \geq 1$ and that equality holds when and only when K is symmetric about O. This functional is in fact an example of a coefficient of asymmetry (see for example (Grünbaum 1963)). In the same paper Sawyer proves

Theorem 3.3 (Sawyer) *Let K be a Λ_n-admissible set in $\mathcal{K}^n$ with coefficient of asymmetry $\lambda(K) = \lambda$. Then*

$$V \leq \phi(\lambda) \det(\Lambda_n),$$

where

$$\phi(\lambda) = (\lambda + 1)^n \left(1 - (1 - \frac{1}{\lambda})^n\right).$$

We note that in the case where K is symmetric about O ($\lambda = 1$), Minkowski's Convex Body Theorem for $\Re^n$ follows immediately from Sawyer's inequality. If, on the other hand, O is the centre of gravity, then $\lambda = 3/2$. In this case, we observe that Ehrhart's conjecture (Conjecture 3.1) gives a much stronger result than Sawyer's theorem. Sawyer (1955a) obtains an exact formula for $\phi(\lambda)$ in the case where $n = 2$ from which the result by Ehrhart (1955a) for $n = 2$ may be deduced. Sawyer (1955b) also obtains estimates for $\phi(\lambda)$ for sets which are symmetric about a point apart from O.

Scott (1974b) obtains a result analogous to the result by Sawyer (1955a) by replacing λ with a certain boundedness condition. We say that a set K is $k\Lambda_n$-bounded if some translate of K is contained in a fundamental cell of $k\Lambda_n$ but no translate of K is contained in any fundamental cell of $(k - \varepsilon)\Lambda_n$ ($\varepsilon > 0$). It is expected that there is a function $f(k)$ for which $V \leq f(k) \det(\Lambda_n)$. Scott finds an exact formulation for $f(k)$ for the case where $n = 2$.

Scott (1978a) also conjectures the following:

Conjecture 3.4 (Scott) *Let K be a Λ_n-admissible set in $\mathcal{K}^n$ and let O_i denote the ith orthant in $\Re^n$ cut off by the coordinate planes. Suppose that $V(K \cap O_i) = 2^{-n}V(K)$. Then*

$$V(K) \leq 2^n \det(\Lambda_n).$$

Scott (1978a) proves the conjecture for the case $n = 2$.

Another conjecture by Scott (1982) concerns the maximal area of a Γ-admissible set having circumcentre O. Using a computer run, we discover that the conjecture is false. We revise the conjecture as follows:

Conjecture 3.5 *Let K be a Γ-admissible set in $\mathcal{K}^2$ having circumcentre O. Then*

$$A \leq \alpha \approx 4.04569.$$

Equality holds when and only when K is congruent to the set shown in Figure 3.6 *(Here $R \approx 1.593$, $\alpha \approx 5.47^o$, $\beta \approx 20.23^o$).*

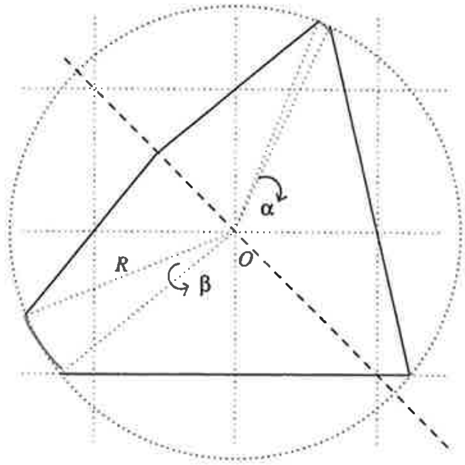


Figure 3.6: The set with maximal area having circumcentre O

Van der Corput (1935, 1936) considers a set K in $\mathcal{K}^n$ with K^o containing more than one point of Λ_n.

Theorem 3.6 (Van der Corput) *Let $K \in \mathcal{K}^n$ and let K be symmetric about O. Suppose that $G(K^o, \Lambda_n) = 2k + 1$. Then*

$$V \leq 2^n(2k + 1)\det(\Lambda_n).$$

Scott (1987) extends Van der Corput's result to the class of non-symmetric convex sets in $\mathcal{K}^2$. Ehrhart (1955c, 1955d) gives partial results relating A and $G(K^o, \Lambda)$ for a set $K \in \mathcal{K}^2$ having centre of gravity at O.

Chapter 4

On the maximal circumradius of a planar convex set containing one lattice point

4.1 Introduction

Let K be a set in $\mathcal{K}^2$ with circumradius $R(K) = R$. A number of results concerning the circumradius of a general convex set are known (Eggleston 1958, p.111; Henk and Tsintsifas 1994; Scott 1978b, 1979a, 1981). However, there are relatively few results on the circumradius of a convex set constrained by lattice points (see for example (Scott 1980)). In this chapter we find the maximal circumradius, under certain conditions, of a Γ-admissible set K where O is the circumcentre of K. It will be seen that the maximal set is a triangle with an edge containing two lattice points.

As in Chapter 3, let S be the closed square with vertices $(\pm 1, \pm 1)$. The eight lattice points $(1,1)$, $(0,1)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, $(1,-1)$, $(1,0)$ on ∂S are labelled $L_1, L_2, \ldots, L_8$ respectively and the half edges, $L_1L_2, L_2L_3, \ldots, L_8L_1$ of S are denoted by h_1, h_2, …, h_8 respectively. Let O be the circumcentre of K and let C be the corresponding circumcircle. We recall from Chapter 3 that K *crosses* an half edge h_i if h_i contains a point in K^o. Suppose that K crosses the half edge h_i. Then h_i partitions K into two regions, one of which does not contain O. Let K' denote this region. If K' intercepts C, we say that K intercepts C beyond the half edge h_i. We prove here the following result concerning R (Awyong and Scott 1995).

Theorem 4.1 *Let K be a Γ-admissible set in $\mathcal{K}^2$ with circumcentre O and circumcircle C and let S denote the square with vertices $(\pm 1, \pm 1)$. Then*

$$R \leq \alpha \approx 1.685,$$

unless K intercepts C beyond exactly two opposite half edges of S, in which case no upper bound can be found for R. Equality holds when and only when K is the triangle shown in Figure 4.1.

The exact value of α is $\alpha = \sqrt{5y^2 + 4y + 1}$ where y is the positive root of $25y^3 + 20y^2 - 3 = 0$.

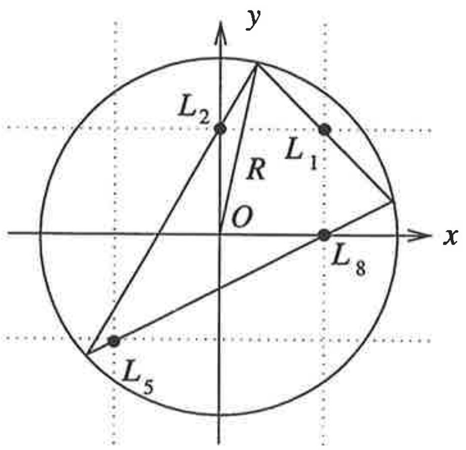


Figure 4.1: A set K for which $R = \alpha \approx 1.685$

4.2 Narrowing the search

If $K = S$, then $R = \sqrt{2} < \alpha$. Hence we may assume that $R > \sqrt{2}$. It follows that K extends beyond ∂S. Convexity arguments show that the convex set K may cross at most four of the eight half edges of S, with no two of the crossed half edges comprising an edge of S. For a given set K, we call the set of half edges beyond which K intercepts C the *intercept set* of K.

It may be proved that either C contains two points of ∂K which are the ends of a diameter of C, or C contains three points of ∂K which form an acute-angled triangle containing O (Yaglom and Boltyanskii 1961, p.59). Henceforth we shall refer to this property of C as the *circumcircle property*. Clearly, by the circumcircle property, K must intercept C beyond at least two half edges of S.

If K intercepts C beyond exactly two half edges of S, using rotations about O and reflections in lines through O to discard equivalent cases, we need only consider the cases where the intercept set of K is one of

$$\text{(a) } \{h_1, h_3\} \quad \text{(b) } \{h_1, h_4\} \quad \text{(c) } \{h_1, h_5\} \quad \text{(d) } \{h_1, h_6\} \quad \text{(e) } \{h_1, h_8\}.$$

In cases (a) (b), (d) and (e), since $O \in K^o$, all intercepts of K and C lie in the half planes $y > 0$, $y > x$, $x > 0$ and $y > 0$ respectively. By the circumcircle property we can discard these cases.

In case (c) a Γ-admissible set K can be constructed having circumcentre O and diameter making a very small angle with the y-axis. It is easily seen that for such a set K, R may be made arbitrarily large. Therefore if K intercepts C beyond exactly two opposite half edges, no upper bound may be found for R.

We may now assume that K intercepts C beyond three or four half edges of S. By the circumcircle property, C contains three points of ∂K which form the vertices of an acute-angled triangle T containing O. In the rest of the chapter, we shall use '*triangle*' to mean a Γ-admissible closed set bounded by the edges of a triangle with circumcentre O. As $R(T) = R(K)$, it is sufficient to establish Theorem 4.1 for the class of triangles. Since T crosses exactly three half edges of S, using rotations and reflections as before, we find that it suffices to consider the cases where the intercept set of T is one of

$$\text{(a) } \{h_1, h_3, h_5\} \quad \text{(b) } \{h_1, h_3, h_8\} \quad \text{(c) } \{h_1, h_4, h_7\} \quad \text{(d) } \{h_1, h_4, h_8\}.$$

In case (b), since $O \in K^o$, all intercepts of K and C lie in the half plane $y > 0$. By the circumcircle property, this case may be eliminated.

We now find an upper bound for $R(T)$ in the cases where T is a triangle having intercept set (a), (c) or (d). Let V_i denote the vertex of T lying beyond h_i and let $\triangle XYZ$ denote the closed triangular region having vertices X, Y and Z. We prove the following lemma:

Lemma 4.2 *Let T be a triangle having intercept set $\{h_1, h_3, h_5\}$, $\{h_1, h_4, h_7\}$ or $\{h_1, h_4, h_8\}$. Then $R(T) < 2$.*

Proof. Let C be the circumcircle of T and let $\text{arc}(X, Y)$ denote the minor arc XY of C, excluding the points X and Y. Let H_i denote the closed half plane not containing the origin and bounded by the line containing the half edge h_i.

Suppose first that T has intercept set $\{h_1, h_3, h_5\}$ (Figure 4.2). Let A and B be the points of intersection of C with the lines $y = 1$ and $y = 0$ respectively in H_3. Let X and Y be the points of intersection of C with the lines $x = -1$ and $x = 0$ respectively in H_5. Since T crosses h_1, h_3 and h_5, we have $V_3 \in \text{arc}(A, B)$ and $V_5 \in \text{arc}(X, Y)$. If $R(T) = 2$, then $A = A^*(-\sqrt{3}, 1)$ and $Y = Y^*(0, -2)$. It is an easy exercise to check that L_4 lies on the same side of A^*Y^* as O. If $R(T) \geq 2$, the line segment AY lies along A^*Y^* or lies on the side of the line A^*Y^* not containing O. Since the edge V_3V_5 lies on the side of AY not containing O, it follows that L_4 lies in the interior of $\triangle OV_3V_5$. Hence $L_4 \in T^o$, contradicting our assumption on T. Hence $R(T) < 2$.

Suppose now that T has intercept set $\{h_1, h_4, h_7\}$ (Figure 4.3). Let A and B be the points of intersection of C with $x = 0$ and $x = 1$ respectively in H_1. Since T crosses h_1, h_4 and h_7, we have $V_1 \in \text{arc}(A, B)$. Since $L_8 \notin T^o$ and the edge V_1V_7 intercepts h_7, it follows that $V_7 \in \text{arc}(X, Y)$, where X and Y are the points of intersection of C with the lines AL_8 and BL_7 respectively in H_7. If $R(T) = 2$, then $A = A^*(0, 2)$ and $X = X^*(\frac{8}{5}, -\frac{6}{5})$. Since T crosses h_1, h_4 and h_7, we also have $V_4 \in \text{arc}(P^*, Q^*)$ where P^* and Q^* are the points with coordinates $(-\sqrt{3}, -1)$ and $(-2, 0)$ respectively. It is easily checked that L_7 lies on the line

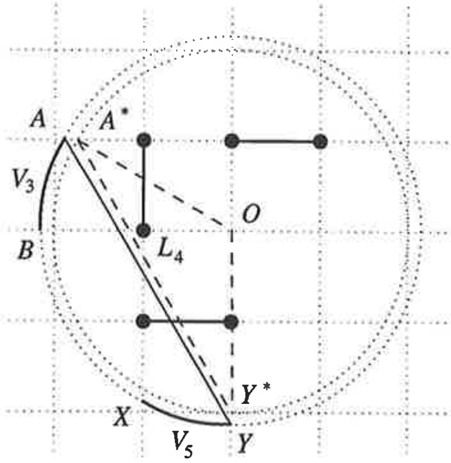


Figure 4.2: The upper bound for $R(T)$ for the $\{h_1, h_3, h_5\}$ case

segment X^*Q^*. Since the edge V_4V_7 lies on the side of X^*Q^* not containing O, it follows that $L_7 \in T^o$, contradicting our assumption on T. If $R(T) > 2$, the line AX is oriented clockwise from the line A^*X^*. Since T crosses h_1, h_4 and h_7, we have $V_4 \in \text{arc}(P, Q)$, where P and Q are the points of intersection of the lines $y = -1$ and $y = 0$ respectively in H_4. It follows that the line segment XQ lies on the side of X^*Q^* not containing O. Since the edge V_4V_7 lies on the side of XQ not containing O, it follows that $L_7 \in T^o$, contradicting our assumption on T. Hence $R(T) < 2$.

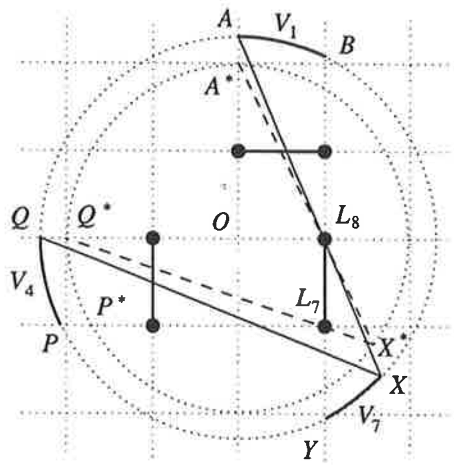


Figure 4.3: The upper bound for $R(T)$ for the $\{h_1, h_4, h_7\}$ case

Finally suppose that T has intercept set $\{h_1, h_4, h_8\}$ (Figure 4.4). Let A and B be the points of intersection of C with the lines $x = 0$ and $x = 1$ in H_1. Let X and Y be the points of intersection of C with the lines $y = 1$ and $y = 0$ respectively in H_8. Since T crosses h_1, h_4 and h_8, we have $V_1 \in \text{arc}(A, B)$ and $V_8 \in \text{arc}(X, Y)$. If $R(T) = 2$, $A = A^*(0, 2)$ and $Y = Y^*(2, 0)$. Clearly, L_1 lies on the line segment A^*Y^*. If $R(T) \geq 2$, the line segment AY lies along A^*Y^* or lies on the side of A^*Y^* not containing O. Since the edge V_1V_8 lies on the side of AY not containing O, it follows that L_1 lies in the interior of $\triangle OV_1V_8$. Hence $L_1 \in T^o$, contradicting our assumption on T. It follows that $R(T) < 2$.

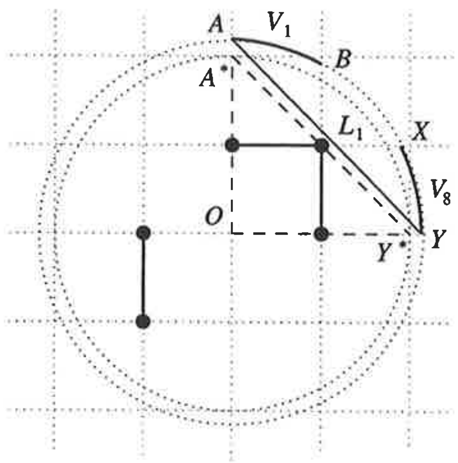


Figure 4.4: The upper bound for $R(T)$ for the $\{h_1, h_4, h_8\}$ case

Hence if T has intercept set $\{h_1, h_3, h_5\}$, $\{h_1, h_4, h_7\}$ or $\{h_1, h_4, h_8\}$, then $R(T) < 2$ and the lemma is proved. □

We now let F denote the family of triangles having given intercept set $\{h_1, h_3, h_5\}$, $\{h_1, h_4, h_7\}$ or $\{h_1, h_4, h_8\}$. By Lemma 4.2, the members of F are contained in a disk of radius 2. By Blaschke's Selection Theorem, F is a compact subset of $\mathcal{K}^2$ in each of these cases, and by Theorem 2.5, a maximal set occurs in each case. Let $\mathcal{T}$ denote a maximal set in F. For a given intercept set, a set K with $R(K) = R(\mathcal{T})$ is referred to as a *maximal* set; in particular, a triangle T with

$R(T) = R(\mathcal{T})$ is referred to as a *maximal* triangle.

In §4.3 we establish some properties of a maximal triangle. In §4.4, we prove two lemmas which will further narrow our search for a maximal set. In §4.5, §4.6 and §4.7, we find the maximal set for the class of triangles having given intercept sets $\{h_1, h_3, h_5\}$, $\{h_1, h_4, h_7\}$ and $\{h_1, h_4, h_8\}$ respectively. For each case we also establish the uniqueness of the maximal triangle by employing the results in §4.4 to eliminate all other triangles as possible solutions.

Unless otherwise specified, the vertices of a triangle will be described in an anticlockwise order.

4.3 Properties of a maximal set

The following lemmas establish some properties of a maximal triangle and a maximal set for intercept sets $\{h_1, h_3, h_5\}$, $\{h_1, h_4, h_7\}$ or $\{h_1, h_4, h_8\}$. Let $\mathcal{L}$ denote the set of non-zero lattice points contained in the circumdisk of T. By Lemma 4.2, we have $R(T) < 2$. We also recall that $R(T) > \sqrt{2}$. Hence $\mathcal{L} = \{L_i, i = 1, \ldots, 8\}$.

Lemma 4.3 *If T is a maximal triangle, then each edge of T must contain a point of $\mathcal{L}$ in its interior.*

Proof. Let $T = \triangle XYZ$ be a maximal triangle with edges x, y and z lying opposite the vertices X, Y and Z respectively, and let C denote the circumcircle of T. By Lemma 4.2, we may assume that $R(T) < 2$. We suppose that there is at least one edge of T which does not contain a point of $\mathcal{L}$ in its interior. If edge x (say) contains no point of $\mathcal{L}$ in its interior, we enlarge T about X until x first contains a point of $\mathcal{L}$. Denote this enlarged triangle T_1 with corresponding vertices X_1, Y_1, Z_1 and edges x_1, y_1, z_1. Let K_1 be the set bounded by C and the edges of T_1. Clearly, K_1 is Γ-admissible and $R(K_1) = R(T)$. By construction, K_1 is bounded by two arcs of C and three straight edges. If now y_1 contains no point of $\mathcal{L}$ its interior, a similar enlargement of T_1 about Y_1 results in a triangle T_2. Let K_2 be the set bounded by C and the edges of T_2. By construction, K_2 is Γ-admissible and has three arc boundaries. Since $R(T) < 2$, C contains no lattice points. We now enlarge C about O by a sufficiently small factor to obtain the set C_1 containing no lattice points. Let K^* be the set bounded by the lines containing the straight edges of T_2 and arcs of C_1. Clearly, K^* is Γ-admissible and $R(K^*) > R(K_2) = R(T)$. This contradicts our assumption that T is a maximal triangle. If now y_1 contains a point of $\mathcal{L}$ in its interior, a small clockwise rotation about the lattice point (choose the lattice point in $\mathcal{L}$ closest to X_1 if there is more than one point of $\mathcal{L}$ in the interior of y_1) also results in a set with three arc boundaries. By the same argument as before, T is not a maximal triangle, contradicting our assumption. □

Lemma 4.4 *If K is a maximal set, then K is a triangle.*

Proof. We suppose that K is a maximal set which is not a triangle. Then by the circumcircle property, K intercepts C in three points which form an acute-angled triangle T containing O. Since K is a maximal set, T is a maximal triangle. As $K \neq T$, there is an edge e of T whose interior lies in K^o. Hence e contains no point of $\mathcal{L}$ in its interior. By Lemma 4.3, T is not a maximal triangle. Hence K is a triangle. □

If each edge of a triangle contains exactly one point of $\mathcal{L}$ in its interior, we call the join of a vertex to the opposite lattice point a *VL-line.*

Lemma 4.5 *If T is a maximal triangle then either*
(i) *its VL-lines are concurrent or*
(ii) *at least one of the edges of T contains two points of $\mathcal{L}$ in its interior.*

Proof. Let $T = \triangle XYZ$ be a maximal triangle. By Lemma 4.3, each edge of T contains at least one point of $\mathcal{L}$ in its interior. Suppose that each edge of

T contains exactly one point of $\mathcal{L}$ in its interior. We let these points be L_x, L_y and L_z lying in the interior of each of the edges x, y and z respectively. Suppose also that the VL-lines, XL_x, YL_y and ZL_z are not concurrent. We transform the edges of the triangle T by first rotating the edge XY (sliding the endpoints on C) about L_z through a small angle θ to $X'Y'$. This is followed by a rotation in the same direction of the edge YZ about L_x to the edge $Y'Z'$. Finally ZX is rotated in the same direction about L_y to $Z'X''$. We denote the described transformation on the edges of T by $f_c(T,\theta)$ if the edges of T are rotated in a clockwise manner, and by $f_a(T,\theta)$, if the edges of T are rotated in an anticlockwise direction. We note that $\angle YXX' = \angle X'Y'Y$ since these two angles are in the same segment of C subtended by chord $X'Y$. Similarly, $\angle Y'YX = \angle Y'X'X$. We therefore deduce that $\triangle YL_zY'$ and $\triangle X'L_zX$ are similar. Hence

$$\frac{XX'}{YY'} = \frac{X'L_z}{L_zY}.$$

Since X, Y and Z are oriented anticlockwise and θ is small and $\triangle XYZ$ contains O in its interior, $f_a(T,\theta)$ gives $X'L_z = XL_z - \varepsilon$. where ε is small and positive. Therefore

$$\frac{XX'}{YY'} = \frac{XL_z - \varepsilon}{L_zY} < \frac{XL_z}{L_zY}.$$

Similarly,

$$\frac{YY'}{ZZ'} < \frac{YL_x}{L_xZ}$$

and

$$\frac{ZZ'}{XX''} < \frac{ZL_y}{L_yX}.$$

Multiplying all three inequalities, we obtain

$$\frac{XX'}{XX''} < \frac{XL_z}{L_zY}.\frac{YL_x}{L_xZ}.\frac{ZL_y}{L_yX} = p, \quad \text{say},$$

where XX'/XX'' differs from p by a small amount ε_p. Similarly, if $f_c(T,\theta)$ is applied,

$$\frac{XX'}{XX''} > p,$$

where XX'/XX'' differs from p by a small amount ε_p'. We note that by Ceva's theorem (Maxwell 1949, p.90), $p = 1$ if and only if the VL-lines XL_x, YL_y and ZL_z are concurrent. By assumption, $p \neq 1$.

If $p > 1$, $f_a(T, \theta)$ yields $XX'/XX'' = p - \varepsilon_p$. We choose θ so that ε_p is sufficiently small to give $XX'/XX'' > 1$. Since now $XX' > XX''$, the transformation results in a set K bounded by edges $X'Y'$, $Y'Z'$, $Z'X''$ and the arc $X''X'$ (Figure 4.5). By Lemma 4.4, K is not a maximal set. Since $R(K) = R(T)$, it follows that T is not a maximal triangle, contradicting our assumption.

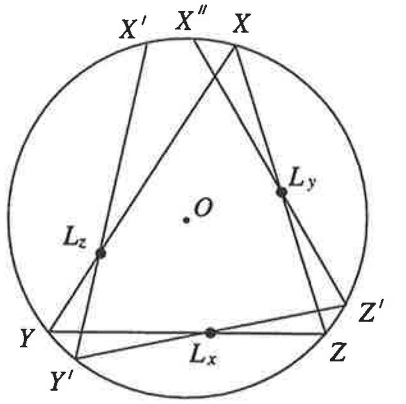


Figure 4.5: The effect of $f_a(T, \theta)$ on T for $p > 1$

If $p < 1$, $f_c(T, \theta)$ yields $XX'/XX'' = p + \varepsilon_p'$. We choose θ so that ε_p' is sufficiently small to give $XX'/XX'' < 1$. Since now $XX' < XX''$, the transformation results in a set K with an arc boundary. Arguing as before, T is not a maximal triangle.

Therefore, the maximal triangle is such that either its VL-lines are concurrent, or it has at least one edge containing two points of $\mathcal{L}$ in its interior. □

For future easy reference, we summarize our findings thus far in the following lemma.

Lemma 4.6 *A maximal set K is a triangle having a point of $\mathcal{L}$ interior to each of its edges and such that either*

(i) *its VL-lines are concurrent or*

(ii) *at least one of its edges contains two points of $\mathcal{L}$ in its interior.*

Proof. By Lemma 4.4, a maximal set K is a triangle. By Lemma 4.3, a maximal triangle T has a point of $\mathcal{L}$ interior to each of its edges and by Lemma 4.5 a maximal triangle T has concurrent VL-lines or has at least one edge containing two points of $\mathcal{L}$. □

4.4 Narrowing the search further

By Lemma 4.6, we may restrict the members of F to those triangles with a given intercept set, and with edges each containing a point of $\mathcal{L}$.

Suppose first that F is the family of triangles having intercept set $\{h_1, h_3, h_5\}$. Let $T \in F$. Since the edges of T each contains a point of $\mathcal{L}$, the edges V_1V_3 and V_3V_5 must contain the points L_2 and L_4 respectively, and the edge V_5V_1 contains L_1 or L_6 or both L_1 and L_6 (Figure 4.6). Table 4.1 gives a list of the possible points of $\mathcal{L}$ contained in the edges V_1V_3, V_3V_5 and V_5V_1.

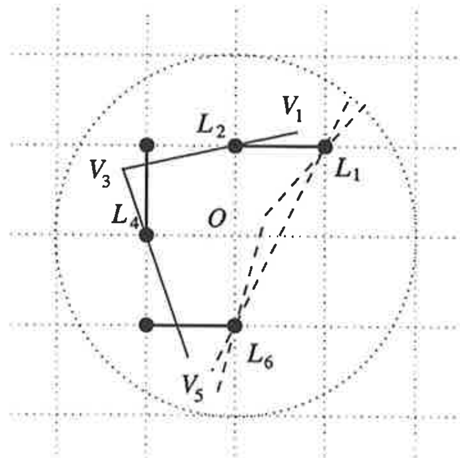


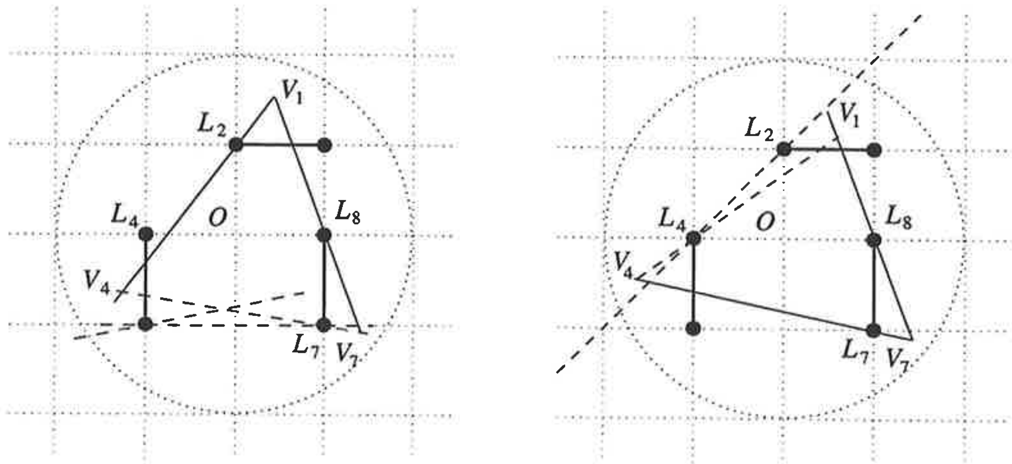
Figure 4.6: Lattice points on the edges of $\triangle V_1V_3V_5$

Suppose now that F is the family of triangles having intercept set $\{h_1, h_4, h_7\}$. Let $T \in F$. We first observe that, given the intercept set of T, the edge V_7V_1

	V_1V_3	V_3V_5	V_5V_1	
(a)	L_2	L_4	L_6	‡
(b)	L_2	L_4	L_1	‡
(c)	L_2	L_4	L_6, L_1	⋆

Table 4.1: Lattice point listings for the $\{h_1, h_3, h_5\}$ case

contains the point L_8. If $m(V_1V_4) > 1$, then V_1V_4 contains the lattice point L_2 and V_4V_7 contains L_5, L_7 or both L_5 and L_7 (Figure 4.7a, cases (a), (b) and (c) respectively of Table 4.2). If on the other hand, $m(V_1V_4) \le 1$, then V_1V_4 contains the lattice point L_4 or both L_2 and L_4 and V_4V_7 contains the lattice point L_7 (Figure 4.7b, cases (d) and (e) of Table 4.2).



(a) $m(V_1V_4) > 1$ (b) $m(V_1V_4) \le 1$

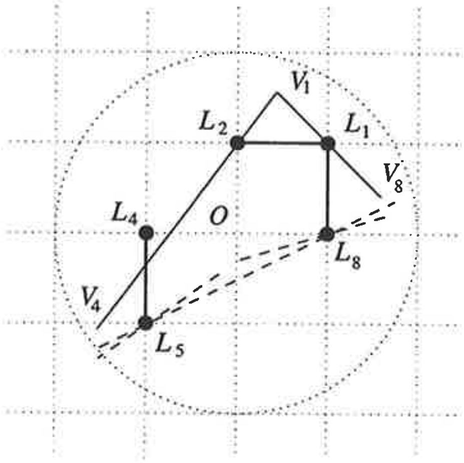
Figure 4.7: Lattice points on the edges of $\Delta V_1V_4V_7$

Finally, let F be the family of triangles having intercept set $\{h_1, h_4, h_8\}$. Let $T \in F$. For this intercept set, the edge V_8V_1 contains the lattice point L_1. If $m(V_1V_4) > 1$, then V_1V_4 contains the point L_2 and V_4V_8 contains the points L_5 or L_8 or both L_5 and L_8 (Figure 4.8a, cases (a), (b) and (c) respectively of Table 4.3). If, on the other hand, $m(V_1V_4) \le 1$, then V_1V_4 contains L_4 or both L_2 and L_4,

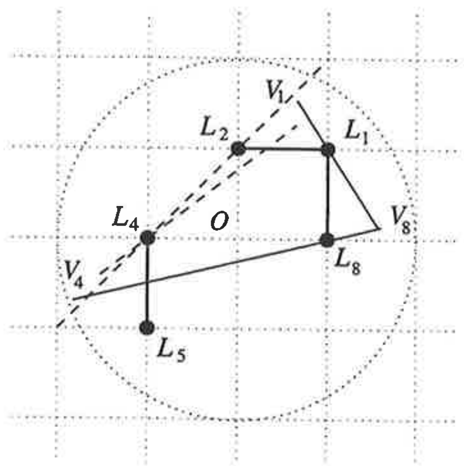
	V_7V_1	V_1V_4	V_4V_7	
(a)	L_8	L_2	L_5	‡
(b)	L_8	L_2	L_7	‡
(c)	L_8	L_2	L_5, L_7	⋆
(d)	L_8	L_4	L_7	‡
(e)	L_8	L_2, L_4	L_7	⋆

Table 4.2: Lattice point listings for the $\{h_1, h_4, h_7\}$ case

and V_4V_8 contains the point L_8 (Figure 4.8b, cases (d) and (e) of Table 4.3).



(a) $m(V_1V_4) > 1$



(b) $m(V_1V_4) \leq 1$

Figure 4.8: Lattice points on the edges of $\triangle V_1V_4V_8$

	V_8V_1	V_1V_4	V_4V_8	
(a)	L_1	L_2	L_5	‡
(b)	L_1	L_2	L_8	‡
(c)	L_1	L_2	L_5, L_8	⋆
(d)	L_1	L_4	L_8	‡
(e)	L_1	L_2, L_4	L_8	⋆

Table 4.3: Lattice point listings for the $\{h_1, h_4, h_8\}$ case

Now to each $T = \triangle XYZ$ in the cases from Tables 4.1, 4.2 and 4.3, we associate the ordered set (ℓ_1, ℓ_2, ℓ_3) called the *lattice-point set* where ℓ_i, $i = 1, 2, 3$ a listing of the points of $\mathcal{L}$ in the interior of the edges YZ, ZX and XY respectively. If T has an edge containing two points of $\mathcal{L}$ in its interior, we label that edge XY and assign the lattice-point set $(L_x, L_y, L_zL'_z)$ where L_z and L'_z appear in the order X, L_z, L'_z, Y.

We will now prove two lemmas to help us narrow down the possibilities for a maximal triangle. Lemma 4.7 establishes the uniqueness of a triangle $T^* = \triangle X^*Y^*Z^*$ with a given lattice-point set $(L_x, L_y, L_zL'_z)$. Therefore, from Tables 4.1, 4.2 and 4.3, it may be seen that there are at most five such triangles (marked $\star$ in the tables) satisfying condition (ii) of Lemma 4.6.

Lemma 4.7 *Let $T = \triangle XYZ$ and $T' = \triangle X'Y'Z'$ be two triangles with the same intercept set and lattice-point set $(L_x, L_y, L_zL'_z)$. Then $T = T'$.*

Proof. Suppose that $T \neq T'$. Then since T and T' have the same lattice point set, we may assume that $R(T) \neq R(T')$. We first suppose that $R(T') > R(T)$. Then $X \in (X', Y)$ and $Y \in (X, Y')$. The edge $X'Z'$ is therefore oriented clockwise about L_y from XZ and the edge $Y'Z'$ is oriented anticlockwise about L_x from YZ. Therefore $Z' \in T^o$ and $OZ' < R(T) < R(T')$. Hence O is not the circumcentre of T', contradicting our assumption on T'. It follows that $R(T') \not> R(T)$. A similar argument shows that $R(T') \not< R(T)$. Therefore $R(T') = R(T)$. Hence $T' = T$ and the lemma is proved. $\square$

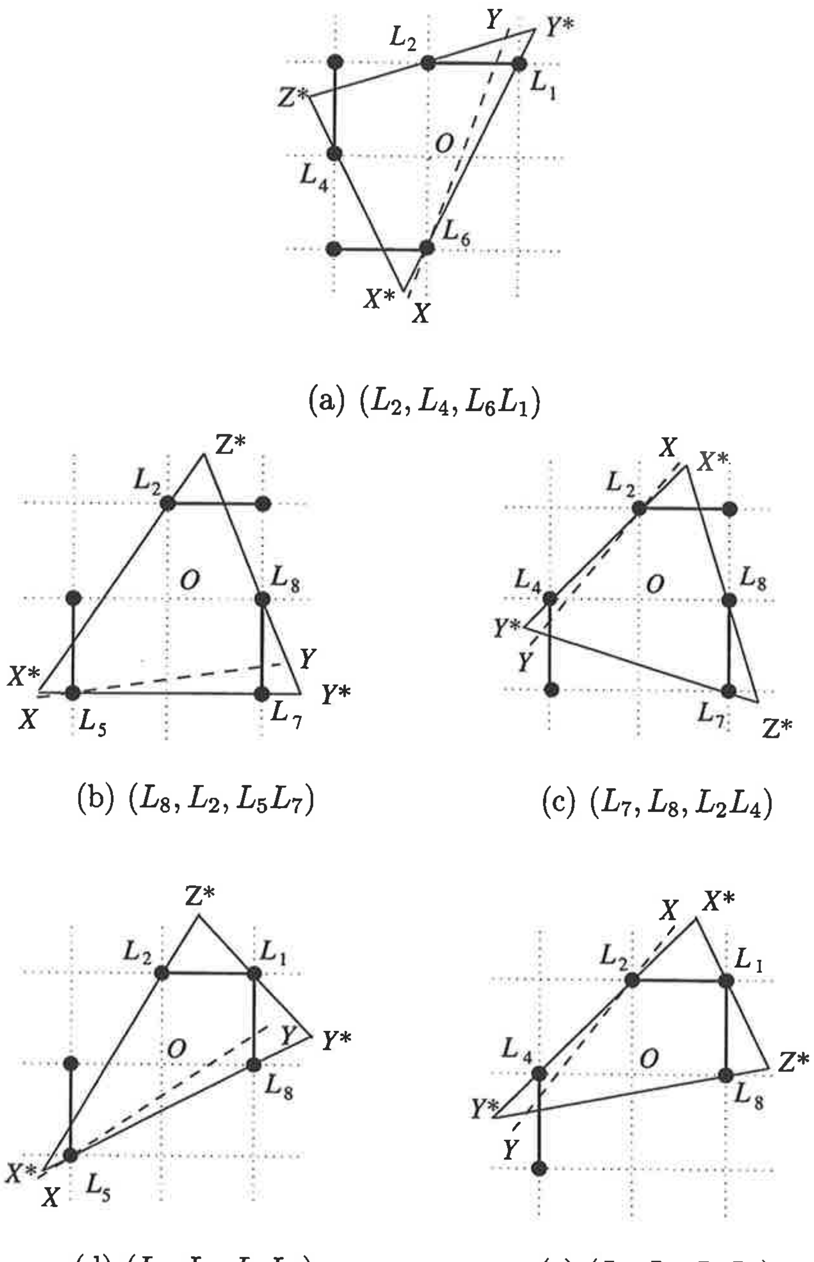
The next lemma helps us to eliminate those cases marked ‡ in the tables. We will show that these triangles have VL-lines which are not concurrent and since these triangles have edges each containing exactly one lattice point, by Lemma 4.6, they are not maximal. We shall be comparing a triangle $T^* = \triangle X^*Y^*Z^*$ along with its given lattice-point set $(L_x, L_y, L_zL'_z)$ with a related triangle $T = \triangle XYZ$ having the same intercept set as T^*.

Lemma 4.8 *Let $T^* = \triangle X^*Y^*Z^*$ denote the unique triangle with a given lattice-point set (L_x, L_y, L_zL_z'). We define P^* to be the intersection of lines X^*L_x and Y^*L_y, and Q^* to be the intersection of lines Z^*P^* and X^*Y^*. If $L_z \in (Q^*, X^*)$, then any triangle with lattice-point set (L_x, L_y, L_z) is not maximal. If $L_z' \in (Q^*, Y^*)$, then any triangle with lattice-point set (L_x, L_y, L_z') is not maximal.*

Proof. Let $T = \triangle XYZ$ be a triangle with the lattice-point set (L_x, L_y, L_z). We define P to be the intersection lines of XL_x and YL_y, and Q to be the intersection of line ZP with XY. We show that line ZL_z cannot pass through P. It will then follow that the VL-lines of T are not concurrent and by Lemma 4.5, T is not a maximal triangle. Let h_x and h_y be the open half planes bounded by the line Q^*Z^* containing X^* and Y^* respectively.

Figure 4.9 shows the five possible triangles T^*, and how the intercept set constrains the edge XY of any triangle $T = \triangle XYZ$ with lattice-point set (L_x, L_y, L_z). Since T is Γ-admissible, T cannot contain L_z' in its interior. Thus in each case, since X^*, L_z, L_z' and Y^* are in the given order, the edge XY of T is oriented anticlockwise about L_z from the edge X^*Y^* of T^*.

We first suppose that T is a maximal triangle. Therefore $R(T) \geq R(T^*)$ which implies that the vertices of T are exterior to T^*. Since the edge XY of T is oriented anticlockwise about L_z from the edge X^*Y^* of T^*, and since the vertices of T are exterior to T^*, the edges YZ and ZX of T are also oriented anticlockwise about L_x and L_y respectively from the corresponding edges of T^* (Figure 4.10). Therefore the VL-lines, XL_x and YL_y of T are oriented anticlockwise about L_x and L_y from X^*L_x and Y^*L_y respectively, placing the point P in the interior of $\triangle Y^*P^*L_x$. It follows that P lies in h_y. Since $L_z \in (Q^*, X^*)$, the lattice point L_z lies in h_x. Also, since the edges YZ and ZX are oriented anticlockwise about L_x and L_y respectively from Y^*Z^* and Z^*X^*, the point Z necessarily lies in h_x. Hence $[Z, L_z]$ is contained in h_x. Therefore $[Z, L_z]$ and the point P are on opposite sides of the line Q^*Z^* and hence the VL-lines of T are not concurrent. Therefore T is not a maximal triangle.



(a) (L_2, L_4, L_6L_1)

(b) (L_8, L_2, L_5L_7)

(c) (L_7, L_8, L_2L_4)

(d) (L_1, L_2, L_5L_8)

(e) (L_8, L_1, L_2L_4)

Figure 4.9: Triangles with edges containing two lattice points

We now let $T = \triangle XYZ$ be a triangle with the lattice-point set (L_x, L_y, L'_z). Arguing in a similar way as above, the edge XY of T is now oriented clockwise

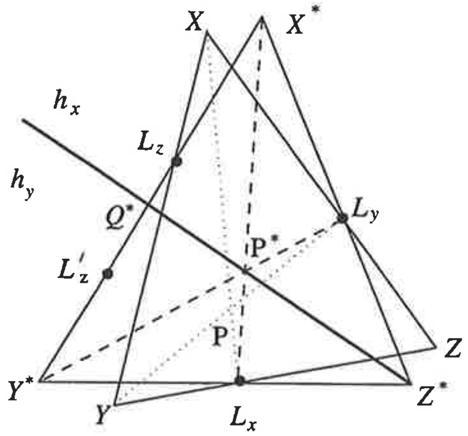


Figure 4.10: The case where $R(T) \geq R(T^*)$

about L'_z from X^*Y^*. Defining P and Q above and using a similar argument, it may be shown that $[Z, L_z]$ and the point P again lie on opposite sides of the line Q^*Z^*. Hence the VL-lines of T are not concurrent and T is not a maximal triangle. □

In the subsequent sections, we will employ Lemma 4.8 to show that the cases marked ‡ are not maximal. Hence we need only consider those cases marked $\star$ in the tables. In other words, a maximal triangle has an edge containing two points of $\mathcal{L}$ in its interior. We will employ the notation used in Lemma 4.8 in the rest of this chapter.

4.5 The $\{h_1, h_3, h_5\}$ case

We first consider case (c) of Table 4.1. Let $T^* = \triangle V_5V_1V_3 \equiv \triangle X^*Y^*Z^*$ be the unique triangle with lattice-point set (L_2, L_4, L_6L_1) (Figure 4.9a).

We assign the coordinates $(x, 2x-1)$ and $(y, 2y-1)$ to X^* and Y^* respectively. Since X^* and Y^* also lie on C,

$$x^2 + (2x-1)^2 = y^2 + (2y-1)^2$$

which gives $x + y = \frac{4}{5}$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^*

contain the lattice points $L_2(0,1)$ and $L_4(-1,0)$ respectively,

$$\frac{z_2-1}{z_1-0}=\frac{2y-2}{y}, \qquad \frac{z_2-0}{z_1+1}=\frac{2x-1}{x+1}.$$

Eliminating x and solving for z_1 and z_2, we obtain

$$z_1=\frac{-y(6+5y)}{25y-18}, \qquad z_2=-\frac{-23y+10y^2+6}{25y-18}.$$

As Z^* also lies on the circumcircle,

$$z_1^2+z_2^2=y^2+(2y-1)^2.$$

Simplifying and factorising, we have

$$(5y-2)(25y^3-45y^2+25y-6)=0.$$

Since $y>1$, we solve $25y^3-45y^2+25y-6=0$ to obtain $y\approx 1.080$ and $R(T^*)\approx 1.584<\alpha$.

We now show that any triangle with lattice-point set (L_2, L_4, L_6) or (L_2, L_4, L_1) (case (a) or case (b) of Table 4.1) is not maximal. We consider the quadrangle $X^*Y^*L_2L_4$. Let K be the point of intersection of the lines X^*Y^* and L_2L_4, that is the point $(2,3)$. By the harmonic property of the quadrangle, the points X^*, Y^* separate Q^* and K harmonically. Therefore the cross-ratio $(X^*, Y^*; Q^*, K)=-1$. We now orthogonally project the four points X^*, Y^*, Q^* and K on the x-axis to obtain the points X_x^*, Y_x^*, Q_x^* and K_x respectively. Since the cross-ratio is unaltered under projection, we have $(X_x^*, Y_x^*; Q_x^*, K_x)=-1$. Letting the x-coordinate of Q^* be q, we have

$$\frac{q-x}{y-q}=-\frac{2-x}{y-2}.$$

which gives $q\approx 0.689$. Therefore $L_6\in(Q^*,X^*)$ and $L_1\in(Q^*,Y^*)$. By Lemma 4.8, any triangle with lattice-point set (L_2, L_4, L_6) or (L_2, L_4, L_1) is not maximal. Hence cases (a) and (b) of Table 4.1 may be eliminated.

Hence $\mathcal{T}$ is the triangle with lattice-point set (L_2, L_4, L_6L_1) and $R(\mathcal{T})\approx 1.584<\alpha$.

4.6 The $\{h_1, h_4, h_7\}$ case

We first consider case (c) of Table 4.2. Let $T^* = \Delta V_4 V_7 V_1 \equiv \Delta X^* Y^* Z^*$ be the triangle with lattice-point set $(L_8, L_2, L_5 L_7)$ (Figure 4.9b).

We assign the coordinates $(1+y, -1)$ to Y^*, $0 < y < 1$. Since X^* Y^* also lie on C, by symmetry, X^* has coordinates $(-1-y, -1)$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^* contain the lattice points $L_8(1,0)$ and $L_2(0,1)$ respectively,

$$\frac{z_2 - 0}{z_1 - 1} = \frac{1}{-y}, \qquad \frac{z_2 - 1}{z_1 - 0} = \frac{2}{1+y}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = \frac{-(y^2 - 1)}{1 + 3y}, \qquad z_2 = \frac{3 + y}{1 + 3y}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = (1 + y)^2 + 1.$$

Simplifying and factorising, we obtain

$$(1 + y)(y^3 + 2y^2 + 2y - 1) = 0.$$

Since $0 < y < 1$, we solve $(y^3 + 2y^2 + 2y - 1) = 0$ for y to obtain $y \approx 0.353$ and $R(T^*) \approx 1.683 < \alpha$.

We now show that any triangle T with lattice-point set (L_8, L_2, L_5) or (L_8, L_2, L_7) (case (a) or case (b) of Table 4.2) is not maximal. We consider the quadrangle $X^*Y^*L_8L_2$. Letting the x-coordinate of Q^* be q and using cross-ratios as in §4.5, we find that $q \approx 0.915$. Therefore $L_5 \in (Q^*, X^*)$ and $L_7 \in (Q^*, Y^*)$. By Lemma 4.8, any triangle with lattice-point set (L_8, L_2, L_5) is (L_8, L_2, L_7) is not maximal. Hence cases (a) and (b) of Table 4.2 may be eliminated.

We now consider case (e) of Table 4.2. Let $T^* = \Delta V_1 V_4 V_7 \equiv \Delta X^* Y^* Z^*$ be the triangle with lattice-point set $(L_7, L_8, L_2 L_4)$ (Figure 4.9c).

We assign the coordinates $(x, x+1)$ to X^*, $0 < x < 1$. Since X^* and Y^* lie on C, by symmetry Y^* has coordinates $(-x-1, -x)$. We let Z^* have coordinates (z_1, z_2). Since Z^*X^* and Y^*Z^* contain the lattice points $L_8(1,0)$ and $L_7(1,-1)$ respectively,

$$\frac{z_2 - 0}{z_1 - 1} = \frac{x+1}{x-1}, \quad \frac{z_2+1}{z_1-1} = \frac{-x+1}{-x-2}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = -\frac{x^2 - 4x - 3}{5x+1}, \quad z_2 = -\frac{x^2+3x+2}{5x+1}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = x^2 + (x+1)^2.$$

Simplifying and factorising, we obtain

$$(2x+1)(2x^3 + 2x^2 - 1) = 0.$$

Since $0 < x < 1$, we solve $2x^3 + 2x^2 - 1 = 0$ to obtain $x \approx 0.565$ and $R(T^*) \approx 1.664 < \alpha$.

We now show that any triangle T with lattice-point set (L_7, L_8, L_4) (case (d) of Table 4.2) is not maximal. We consider the quadrangle $X^*Y^*L_8L_7$. Letting the x-coordinate of Q^* be q and considering cross-ratios as in §4.5, we find that $q \approx 0.256$. Therefore $L_4 \in (Q^*, Y^*)$. By Lemma 4.8, any triangle with lattice-point set (L_7, L_8, L_4) is not maximal. Hence case (d) of Table 4.2 may be eliminated.

4.7 The $\{h_1, h_4, h_8\}$ case

We first consider case (c) of Table 4.3. Let $T^* = \Delta V_4V_8V_1 \equiv \Delta X^*Y^*Z^*$ be the triangle with lattice-point set (L_1, L_2, L_5L_8) (Figure 4.9d).

We assign the coordinates $(2x+1, x)$ and $(2y+1, y)$ to X^* and Y^* respectively. Since X^* and Y^* also lie on C,

$$(2x+1)^2 + x^2 = (2y+1)^2 + y^2$$

which gives $x + y = -\frac{4}{5}$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^* contain the lattice points $L_1(1,1)$ and $L_2(0,1)$ respectively,

$$\frac{z_2 - 1}{z_1 - 1} = \frac{y-1}{2y}, \qquad \frac{z_2 - 1}{z_1 - 0} = \frac{x-1}{2x+1}.$$

Eliminating x and solving for z_1 and z_2 in terms of y, we obtain

$$z_1 = -\frac{-3 - 7y + 10y^2}{25y + 3}, \qquad z_2 = -\frac{-12 - 21y + 5y^2}{25y + 3}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = (2y+1)^2 + y^2.$$

Simplifying and factorising we obtain

$$(5y + 2)(25y^3 + 20y^2 - 3) = 0$$

Since $y > 0$, we solve $25y^3 + 20y^2 - 3 = 0$ for y to obtain $y \approx 0.326$ and $R(T^*) \approx 1.685 = \alpha$.

We now show that any triangle with lattice-point set (L_1, L_2, L_5) and (L_1, L_2, L_8) (case (a) or case (b) of Table 4.3) is not maximal. We consider the quadrangle $X^*Y^*L_1L_2$ and using cross-ratios as before, we show that the x-coordinate q of Q^* is approximately 0.953. Therefore $L_5 \in (Q^*, X^*)$ and $L_8 \in (Q^*, Y^*)$. By Lemma 4.8, any triangle with lattice-point set (L_1, L_2, L_5) and (L_1, L_2, L_8) is not maximal. Hence cases (a) and (b) of Table 4.3 may be eliminated.

We now consider case (e) of Table 4.3. Let $T^* = \triangle V_1V_4V_8 \equiv \triangle X^*Y^*Z^*$ be the triangle with lattice-point set (L_8, L_1, L_2L_4) (Figure 4.9e).

We assign the coordinates $(x, x+1)$ to X^*. Since X^* and Y^* also lie on C, by symmetry, Y^* has coordinates $(-x-1, -x)$. We let Z^* have coordinates (z_1, z_2). Since Z^*X^* and Y^*Z^* contain the lattice points $L_1(1,1)$ and $L_8(1,0)$ respectively,

$$\frac{z_2 - 1}{z_1 - 1} = \frac{x}{x-1}, \qquad \frac{z_2 - 0}{z_1 - 1} = \frac{-x}{-x-2}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = \frac{-x^2+2x+2}{3x}, \qquad z_2 = \frac{-x+1}{3}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = x^2 + (x+1)^2.$$

Simplifying and factorising, we obtain

$$(2x+1)(2x^3+2x^2-1) = 0.$$

Since $0 < x < 1$, we solve $2x^3 + 2x^2 - 1 = 0$ for x to obtain $x \approx 0.565$ and $R(T^*) \approx 1.664 < \alpha$.

We now show that any triangle with lattice-point set (L_8, L_1, L_4) (case (d) of Table 4.3) is not maximal. We consider the quadrangle $X^*Y^*L_8L_1$. Using the cross-ratio argument, we find that the x-coordinate q of Q^* is approximately 0.256 and therefore $L_4 \in (Q^*, Y^*)$. By Lemma 4.8, any triangle with lattice-point set (L_8, L_1, L_4) is not maximal. Hence case (d) of Table 4.3 may be eliminated.

Comparing the results in §4.5, §4.6 and §4.7. we conclude that the maximal set K is the triangle with lattice-point set (L_1, L_2, L_5L_8) (case (c) of Table 4.3) with $R(K) \approx 1.685$. Theorem 4.1 is therefore proved.

4.8 Comment

It is interesting to observe that the triangles with lattice-point sets (L_7, L_8, L_2L_4) in §4.6 and (L_8, L_1, L_2L_4) in §4.7 have the same circumradius. There does not appear to be any obvious algebraic connection, and we have been unable to find a simple geometrical proof.

Chapter 5

On the width of a planar convex set containing zero, one or two lattice points

5.1 Introduction

Let K be a set in $\mathcal{K}^2$ with width $w(K) = w$. Scott (1973) shows that if $G(K^o, \Gamma) = 0$, then

$$w \leq \frac{1}{2}(2 + \sqrt{3}), \tag{5.1}$$

with equality when and only when $K \cong \mathcal{E}_0$, where $\mathcal{E}_0$ is the equilateral triangle of Figure 5.1.

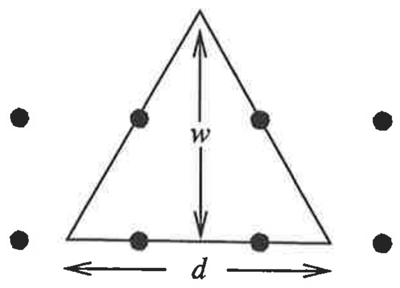


Figure 5.1: The equilateral triangle $\mathcal{E}_0$

Elkington and Hammer (1976) make use of the value $\lambda = \frac{1}{2}(2+\sqrt{3})$ to obtain the result that if $w > r\lambda$ for $r \in \mathbf{Z}$, then $G(K^o, \Gamma) \geq r^2$. It may therefore be deduced that if $G(K^o, \Gamma) < 2^2$, then $w \leq 2\lambda = 2 + \sqrt{3}$. Scott (1985a) sharpens this inequality for a set K with $G(K^o, \Gamma) = 1$. In this case,

$$w \leq 1 + \sqrt{2}, \tag{5.2}$$

with equality when and only when K is congruent to the isosceles triangle shown in Figure 5.2.

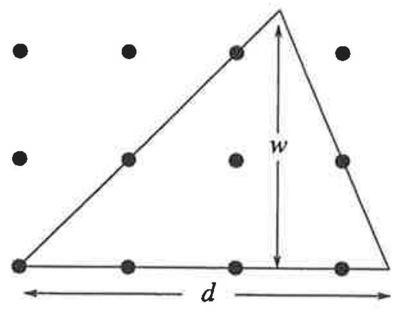


Figure 5.2: The set with maximal width for $G(K^o, \Gamma) = 1$

In this chapter we find a best upper bound for w in the case where $G(K^o, \Gamma) = 2$. We will see that the result follows easily by generalizing (5.1) to the rectangular lattice. Let $\Lambda_R(\mathbf{u},\mathbf{v})$ denote the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$ where $u \leq v$. We prove

Theorem 5.1 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$w \leq \frac{1}{2}(\sqrt{3}u + 2v), \tag{5.3}$$

with equality when and only when $K \cong \mathcal{E}_R$, where $\mathcal{E}_R$ is the equilateral triangle of Figure 5.3.

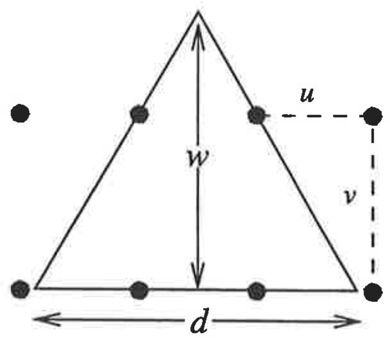


Figure 5.3: The equilateral triangle $\mathcal{E}_R$

Theorem 5.1 has in fact been proved in the special case where $v \leq \sqrt{3}u$ (Scott 1993). We also note that Theorem 5.1 follows from a recent generalization of

(5.1) to arbitrary lattices by Vassallo (1992). As Vassallo's proof is quite long and involved, we give here a short proof for Theorem 5.1 by adapting the methods by Scott (1973, 1993). We will then use Theorem 5.1 to find a best upper bound for $w(K)$ in the case where $G(K^o, \Gamma) = 2$. We prove the following result:

Theorem 5.2 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 2$. Then*

$$w \leq \frac{1}{2}(4 + \sqrt{3}). \tag{5.4}$$

Equality occurs when and only when $K \cong \mathcal{E}_2$, where $\mathcal{E}_2$ is the equilateral triangle of Figure 5.4.

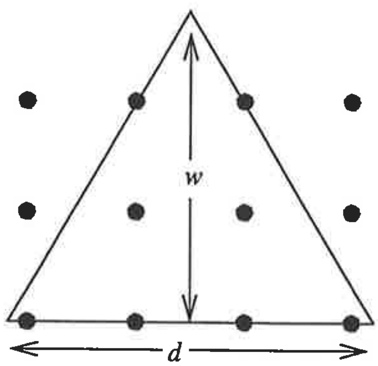


Figure 5.4: The equilateral triangle $\mathcal{E}_2$

5.2 Two useful results

We first establish two useful results.

Lemma 5.3 *Let K be a set in $\mathcal{K}^2$ contained in a triangle T satisfying the conditions of the Triangle Rotation Lemma. Then there is a triangle $\triangle$ circumscribing a cell of Λ_R with $w(K) \leq w(\triangle)$. Equality holds when and only when $K \cong \triangle$.*

Proof. Since $K \subseteq T$, we have $w(K) \leq w(T)$. By the Triangle Rotation Lemma, either T circumscribes a cell of Λ_R or there is a triangle $T_* \cong \lambda T$, $\lambda > 1$, with T_* circumscribing a cell of Λ_R. In either case, there is a triangle $\triangle$

circumscribing a cell of Λ_R with $w(T) \leq w(\triangle)$. It follows that $w(K) \leq w(\triangle)$ with equality when and only when $K \cong \triangle$. □

Lemma 5.4 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Suppose that $w > v$ and $r > \frac{1}{2}u$. Then $K \subseteq T$ where T is a triangle satisfying the Triangle Rotation Lemma or $K \subseteq Q$ where Q is a proper convex quadrilateral circumscribing a cell of Λ_R with edges each containing in its interior a vertex of the cell.*

Proof. We will use the notation employed in the Triangle Rotation Lemma. Thus x is a line containing the lattice point X and $d(P, x)$ denotes the distance from a point P to the line x. In addition we let m_x denote the slope of the line x and we write $T(c, d, x)$ to denote a triangle determined by the lines c, d and x satisfying the conditions of the Triangle Rotation Lemma. Let $\mathcal{D}$ denote an indisk of K. We first translate K through a suitable lattice vector so that the centre of $\mathcal{D}$ lies in $0 < y < v$. For later reference, we list the above properties of $\mathcal{D}$ as follows:

P1. $r > \frac{1}{2}u$.

P2. The centre of $\mathcal{D}$ lies in $0 < y < v$.

Now since $w > v$, K must extend beyond at least one of the lines $y = 0$ and $y = v$. Without loss of generality, suppose that K extends beyond the line $y = v$. Then since $G(K^o, \Lambda_R) = 0$, K^o intercepts $y = v$ between two adjacent lattice points on $y = v$. By translating K through a suitable lattice vector we may take these points to be $G(u, v)$ and $H(0, v)$. Hence K is bounded by lines g and h, with $m_g \neq 0$ and $m_h \neq 0$. By P1 and P2, g and h intersect in the halfplane $y > v$.

Let E and F be the points $(0, 0)$ and $(u, 0)$ respectively and let $\mathcal{C}$ denote the closed cell $EFGH$. From P1 and P2, we deduce that K^o intercepts at least one of the edges FG and EH of $\mathcal{C}$. Without losing generality, we may assume that

K^o intercepts the edge FG of $\mathcal{C}$. Therefore K lies below a line g with $m_g < 0$. We now consider the following two cases:

Case 1: K is bounded by $y = 0$ or K crosses $y = 0$ between the points E and F. Here we note that $m_f \geq 0$. We have the following possibilities:

(a) $m_f = 0$ and $m_h \neq 0$. Hence $K \subseteq T(g, h, x)$ where x is a line containing the lattice point X on the edge of T lying on $y = 0$ (Figure 5.5a).

(b) $m_f > 0$ and ($m_h < 0$ (possibly infinite) or $m_e > 0$). Hence $K \subseteq T(g, f, h)$ or $K \subseteq T(f, g, e)$ (Figure 5.5b).

(c) $m_f > 0$ and ($m_h > 0$ and $m_e < 0$). Hence $K \subseteq Q$ where Q is a proper convex quadrilateral with edges each containing in its interior a vertex of $\mathcal{C}$ (Figure 5.5c).

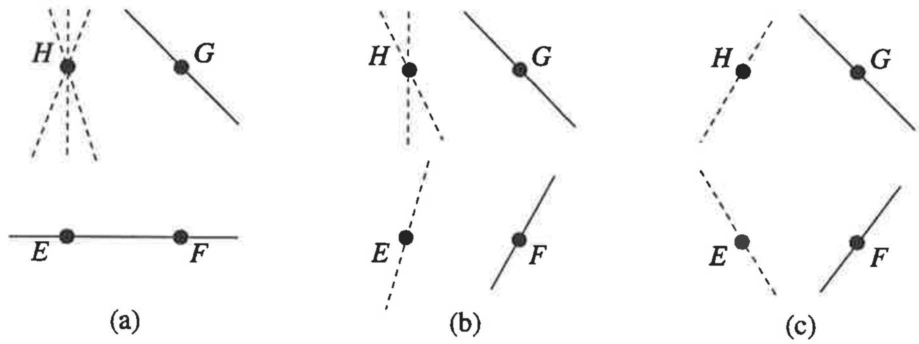


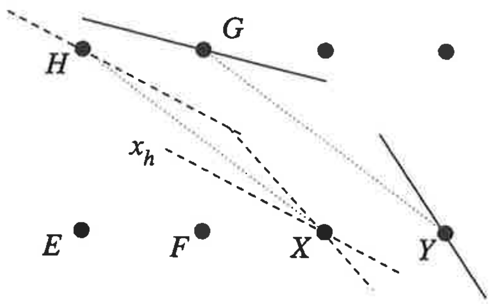
Figure 5.5: Case 1 of Lemma 5.4

Case 2: K crosses the line $y = 0$ between adjacent lattice points $X(ku, 0)$ and $Y((k+1)u, 0)$ for some non-zero integer k. By a reflection of the set K in the mediator of the line segment EF, we may assume that $k > 0$. Here K is bounded by lines x and y with $m_x < 0$. By P1 and P2, the lines x and y intersect in $y < 0$. Furthermore, by P1 and P2, K^o intercepts at least one of the edges HX and YG of the parallelogram $XYGH$. Without loss of generality, let K^o intercept the edge YG having slope m. Hence $m_g > m$ and $m_y < m$. We have the following possibilities:

(a) $m_x \leq m$ or $m_h \geq m$ (Figure 5.6a). If $m_x \leq m$, then we have $K \subseteq T(x, y, g)$. If $m_h \geq m$, then $K \subseteq T(x_h, y, g)$ where x_h is a line containing the point X and parallel to the line h.

(b) $m_h < m$ and $m_x > m$ (Figure 5.6b). If $m_g > m_x$, then we have $K \subseteq T(x, y, g)$. If $m_g \leq m_x$ (an infinite triangle in the case where $m_g = m_x$), then we have $K \subseteq T(g, h, x)$.

(a)



(b)

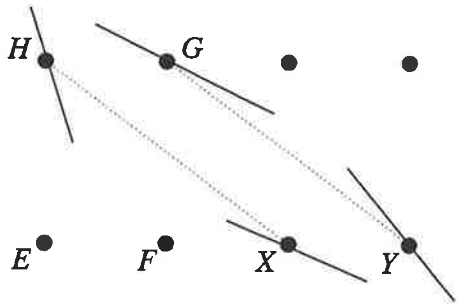


Figure 5.6: Case 2 of Lemma 5.4

From the cases enumerated above, it follows that either $K \subseteq T$ where T is a triangle satisfying the conditions of the Triangle Rotation Lemma or $K \subseteq Q$ where Q is a proper convex quadrilateral with edges each containing in its interior a vertex of a cell of Λ_R. □

5.3 Proof of Theorem 5.1

Let K now be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$ and for which w is as large as possible. From the equilateral triangle $\mathcal{E}_R$ (Figure 5.3) we may assume that $w \geq \frac{1}{2}(\sqrt{3}u + 2v) > v$. By a well known result of Blaschke (Yaglom and Boltyanskii

1961, p.18), K contains a disk $\mathcal{D}$ of radius $r = \frac{1}{3}.\frac{1}{2}(\sqrt{3}u + 2v)$. Since $v \geq u$, we have

$$r = \frac{1}{3}.\frac{1}{2}(\sqrt{3}u + 2v) \geq \frac{1}{3}.\frac{1}{2}(\sqrt{3} + 2)u > \frac{u}{2}.$$

Since $w > v$ and $r > \frac{1}{2}u$, by Lemma 5.4, $K \subseteq T$ where T is a triangle satisfying the conditions of the Triangle Rotation Lemma or $K \subseteq Q$ where Q is a proper, convex quadrilateral as described in Lemma 5.4. If $K \subseteq T$, by Lemma 5.3, K is a triangle circumscribing a cell of Λ_R. It follows that K is a closed convex quadrilateral (possibly degenerate) circumscribing a cell of Λ_R.

Since the constraint $v \leq \sqrt{3}u$ is used by Scott (1993) only to show the existence of the quadrilateral, we can use the remainder of the proof given there to obtain (5.3) with equality when and only when $K \cong \mathcal{E}_R$, where $\mathcal{E}_R$ is the equilateral triangle of Figure 5.3. Hence Theorem 5.1 is proved.

5.4 Proof of Theorem 5.2

Let K now be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 2$. By $SL2$ (see §2.5), we have the following two cases:

(i) If z_1 and z_2 are both odd, we consider the sublattice Γ' and observe that $G(K^o, \Gamma') = 0$. In this case $u = v = \sqrt{2}$ and by Theorem 5.1, we have

$$w \leq \frac{1}{2}(\sqrt{3}\sqrt{2} + 2\sqrt{2}) < 2 + \frac{\sqrt{3}}{2}.$$

(ii) If say z_1 is odd and z_2 is even, we consider the sublattice Γ'' and observe that $G(K^o, \Gamma'') = 0$. Here $u = 1$, $v = 2$ and by Theorem 5.1, we have

$$w \leq \frac{1}{2}(\sqrt{3}.1 + 2.2) = 2 + \frac{\sqrt{3}}{2}.$$

Equality occurs when and only when $K \cong \mathcal{E}_2$ (Figure 5.4). This completes the proof of Theorem 5.2.

5.5 Other related results

We note that (5.1) may be deduced directly from the following result by Eggleston (1959):

Theorem 5.5 (Eggleston) *Let K be a set in $\mathcal{K}^2$ and let S be a square of side length s inscribed in K. Then*

$$w \leq \frac{1}{2}(2+\sqrt{3})s.$$

Equality holds when and only when K is an equilateral triangle.

McMullen and Wills (1981) generalize (5.1) to $\Re^n$ as follows:

Theorem 5.6 (McMullen and Wills) *Let*

$$\triangle_n = \max\{w(K) : K \in \mathcal{K}^n, G(K^o, \Gamma_n) = 0\}.$$

Then

$$(\sqrt{2}+1)(\sqrt{(n+1)} - \beta) < \triangle_n \leq \begin{cases} n, & n \text{ odd}, \\ (n+1)\sqrt{n(n+2)^{-1}}, & n \text{ even}, \end{cases}$$

where

$$\beta = 3\sqrt{2} - 4 + 7\sqrt{\frac{1}{3}} - 4\sqrt{\frac{2}{3}} \approx 1.018.$$

By using the idea of a Minkowski-reduced basis, Vassallo (1992) extends (5.1) to the general lattice Λ. A basis $\{\mathbf{a}, \mathbf{b}\}$ is a Minkowski-reduced basis for a lattice Λ if

(a) $\mathbf{a} \in \{\mathbf{u} \in \Lambda\backslash\{O\}; |\mathbf{u}| \text{ is minimal}\}$

(b) $\mathbf{b} \in \{\mathbf{v} \in \Lambda\backslash\{O\};\ \{\mathbf{a},\mathbf{b}\} \text{ is a basis for } \Lambda \text{ and } |\mathbf{v}| \text{ is minimal}\}$

(c) $\mathbf{a}.\mathbf{b} \geq 0$.

Vassallo (1992) proves

Theorem 5.7 (Vassallo) *Let* $\{\mathbf{a}, \mathbf{b}\}$ *be a Minkowski-reduced basis for the lattice* Λ *in* $\Re^2$ *and let* θ *be the angle between* $\mathbf{a}$ *and* $\mathbf{b}$. *Let* K *be a set in* $\mathcal{K}^2$ *with* $G(K^o, \Lambda) = 0$ *Then*

$$w \leq \frac{\sqrt{3}}{2}|\mathbf{a}| + |\mathbf{b}| \sin\theta.$$

Equality is attained when and only when K *is congruent to the equilateral triangle shown in* Figure 5.7.

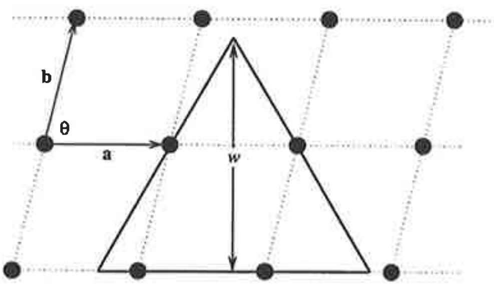


Figure 5.7: The set with maximal width for the case $G(K^o, \Lambda) = 0$

The next two results concern the equilateral triangle lattice. Let Λ_T denote the equilateral triangle lattice generated by the vectors $(1,0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. Scott (1978c) proves

Theorem 5.8 (Scott) *Let* $K \in \mathcal{K}^2$ *and let* $G(K^o, \Lambda_T) = 0$. *Then*

$$w \leq \sqrt{3}.$$

Equality holds when and only when K *is congruent to the equilateral triangle of side length* 2 (Figure 5.8).

Scott's result above in fact follows immediately from Theorem 5.7 by taking $\mathbf{a} = (1,0)$ and $\mathbf{b} = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$. Wetwitschka (1991) extends Scott's result above to sets K having $G(K^o, \Lambda_T) = 1$.

Theorem 5.9 (Wetwitschka) *Let* $K \in \mathcal{K}^2$ *and let* $G(K^o, \Lambda_T) = 1$. *Then*

$$w \leq \frac{3}{2}\sqrt{3},$$

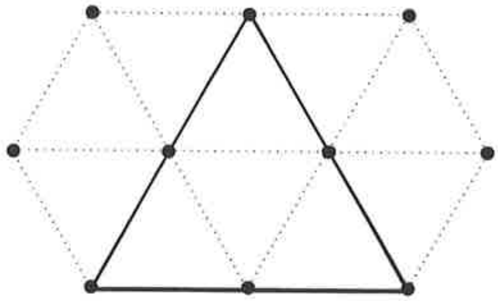


Figure 5.8: The set with maximal width for the case $G(K^o, \Lambda_T) = 0$

with equality when and only when K is congruent to the equilateral triangle shown in Figure 5.9.

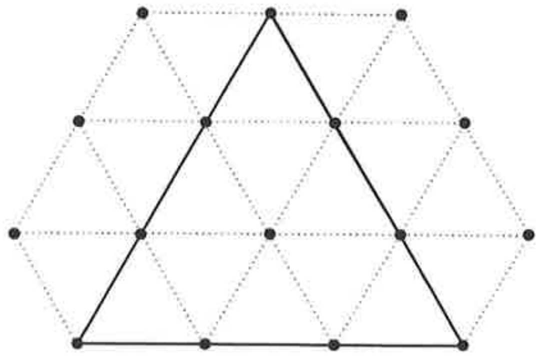


Figure 5.9: The set with maximal width for the case $G(K^o, \Lambda_T) = 1$

A number of results concerning the width of sets in $\mathcal{K}^2$ with special properties have been obtained. Sallee (1969) obtains the following result concerning the maximal width of a set of constant width.

Theorem 5.10 (Sallee) *Let K be a set of constant width in $\mathcal{K}^2$ and let $G(K^o, \Gamma) = 0$. Then*

$$w \leq \alpha \approx 1.545,$$

where α is the root of

$$2x^4 + x^3(2\sqrt{3} - 1) + x^2(-2 - \sqrt{3}) + x(-1 - 3\sqrt{3}) - 2 = 0.$$

Equality is attained when and only when K is the Reuleaux triangle shown in Figure 5.10.

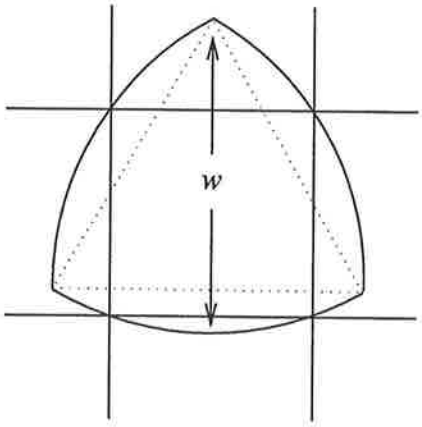


Figure 5.10: The Reuleaux triangle

Scott (1982) conjectures the following result:

Conjecture 5.11 (Scott) *Let K be a Γ-admissible set in $\mathcal{K}^2$ and let O be the centre of gravity of K. Then*

$$w \leq \frac{3}{2}\sqrt{2}.$$

Equality holds when and only when K is congruent to Ehrhart's triangle (Figure 3.1).

Chapter 6

Width-diameter relations for convex sets with lattice point constraints in the plane

6.1 Introduction

Let K be a set in $\mathcal{K}^2$ with width $w(K) = w$ and diameter $d(K) = d$. Scott (1979b) proves that if $G(K^o, \Gamma) = 0$, then

$$(w-1)(d-1) \leq 1, \tag{6.1}$$

with equality when and only when $K \cong \mathcal{T}_0$, where $\mathcal{T}_0$ is the triangle of Figure 6.1.

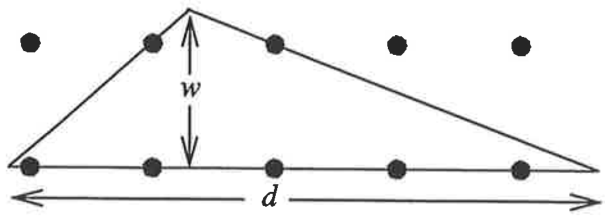


Figure 6.1: The triangle $\mathcal{T}_0$

This result is extended to the case $G(K^o, \Gamma) = 1$ (Scott 1985a) where we have

$$(w-\sqrt{2})(d-\sqrt{2}) \leq 2, \tag{6.2}$$

with equality when and only when $K \cong \mathcal{T}_1$, where $\mathcal{T}_1$ is the triangle of Figure 6.2.

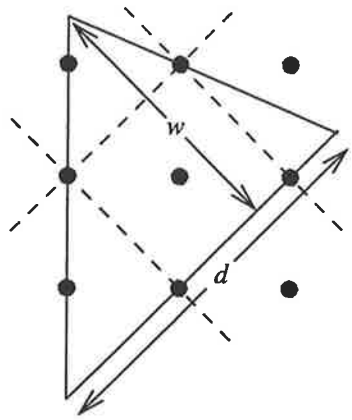


Figure 6.2: The triangle $\mathcal{T}_1$

In this chapter, we generalize (6.1) to rectangular lattices (Awyong and Scott 1996a). Let $\Lambda_R(\mathbf{u},\mathbf{v})$ denote the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$ where $u \leq v$. We prove

Theorem 6.1 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$(w - v)(d - u) \leq uv, \tag{6.3}$$

with equality when and only when $K \cong \mathcal{T}_R$, where $\mathcal{T}_R$ is the triangle of Figure 6.3.

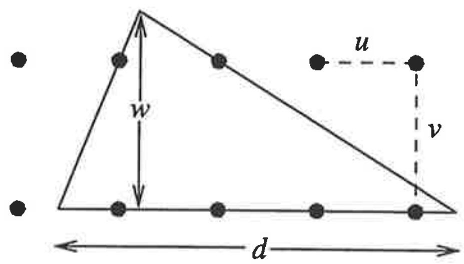


Figure 6.3: The triangle $\mathcal{T}_R$

We then use this result to obtain a corresponding inequality for the case $G(K^o, \Gamma) = 2$. We prove

Theorem 6.2 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 2$. Then*

$$(w - 2)(d - 1) \leq 2, \tag{6.4}$$

with equality when and only when $K \cong \mathcal{T}_2$, where $\mathcal{T}_2$ is the triangle of Figure 6.4.

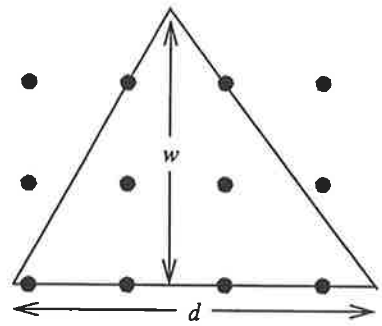


Figure 6.4: The triangle $\mathcal{T}_2$

6.2 Three useful lemmas

Suppose that $K \in \mathcal{K}^2$ and $G(K^o, \Lambda_R) = 0$. Let

$$f(K) = (w(K) - v)(d(K) - u) = (w - v)(d - u).$$

Clearly we may assume that $d \geq w > v \geq u$. We aim to find the maximal value of $f(K)$. We first establish three lemmas which will help us narrow the possibilities for a maximal set.

In Lemma 6.3, we establish the maximal value of $f(K)$ for the class of triangles circumscribing a cell of Λ_R. Lemmas 6.4 and 6.5 will help us eliminate those sets which are not maximal.

Lemma 6.3 *Let K be a triangle circumscribing a cell of Λ_R. Then*

$$f(K) = (w - v)(d - u) \leq uv,$$

with equality when and only when the edge of K with length d lies on the side of the cell having length u.

Proof. Let the vertices of K be X, Y and Z and let C denote the cell inscribed in K. Without loss of generality, let XY be the side of K containing two vertices of C. Let XY have length b and let the altitude from Z to XY be h.

We first let the edge XY lie on the edge of C with length u. Then $A = \frac{1}{2}bh(= \frac{1}{2}wd)$. The edges of C partition K into four regions. The area of K may therefore be calculated as the sum of the areas of the four component parts (Figure 6.5).

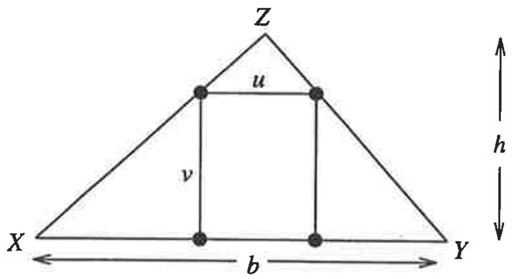


Figure 6.5: A triangle circumscribing a cell of Λ_R

Hence

$$\begin{aligned} \frac{1}{2}wd = \frac{1}{2}bh &= \frac{1}{2}(b-u)v + \frac{1}{2}(h-v)u + uv \\ &= \frac{1}{2}(bv + hu), \end{aligned}$$

that is,

$$wd = bh = bv + hu.$$

We first note that

$$(bv + hu)^2 = (bv - hu)^2 + 4(bv)(hu).$$

Since $wd = bh$, we have $(bv)(hu) = (dv)(wu)$. Furthermore, since $0 < (bv - hu) \leq (dv - wu)$, it follows that

$$(bv + hu)^2 \leq (dv - wu)^2 + 4(dv)(wu) = (dv + wu)^2.$$

Hence

$$bv + hu \leq dv + wu.$$

We thus have

$$wd \leq dv + wu.$$

Adding uv to both sides of the inequality gives

$$f(K) = (w - v)(d - u) \leq uv.$$

Equality is attained here when $XY = b = d$ and $h = w$.

If, on the other hand, XY lies on the edge of $\mathcal{C}$ of length v (here we may assume that $u \neq v$, otherwise we have the above case), then we obtain $(w-u)(d-v) \leq uv$. In this case we write

$$f(K) = (w - v)(d - u) = (w - u)(d - v) + (w - d)(v - u) \tag{6.5}$$

Since $w < d$ for triangles, and $u < v$, we have

$$(w - v)(d - u) < (w - u)(d - v) \leq uv.$$

Hence $f(K) \leq uv$ with equality when and only when the edge of K with length d lies on the side of the inscribed cell of length u. □

Lemma 6.4 *Let K be a set in $\mathcal{K}^2$ contained in a triangle T satisfying the conditions of the Triangle Rotation Lemma. Then there is a triangle $\triangle$ circumscribing a cell of Λ_R with $f(K) \leq f(\triangle)$. Equality holds when and only when $K \cong \triangle$.*

Proof. Since $K \subseteq T$, $f(K) \leq f(T)$. By the Triangle Rotation Lemma, either T circumscribes a cell of Λ_R or there is a triangle $T_* \cong \lambda T$, $\lambda > 1$, with T_* circumscribing a cell of Λ_R. In either case, there is a triangle $\triangle$ circumscribing a cell of Λ_R with $f(T) \leq f(\triangle)$. It follows that $f(K) \leq f(\triangle)$ with equality when and only when $K \cong \triangle$. □

Lemma 6.5 *Let Q be a proper convex quadrilateral circumscribing a cell of Λ_R and having edges each containing in its interior a vertex of the cell. If K is contained in Q, then $f(K) < uv$.*

Proof. We shall denote lines with lower case letters. Thus x is a line containing the point X. Let $d(X, y)$ denote the distance of point X from line y. Let the vertices of the circumscribed cell of Λ_R be A, B, C and D labelled anticlockwise and let $Q = XYZW$ where $a.b = X$, $b.c = Y$, $c.d = Z$ and $d.a = W$ (Figure 6.6). Since $K \subseteq Q$, we have $f(K) \leq f(Q)$. It therefore suffices to show that $f(Q) < uv$.

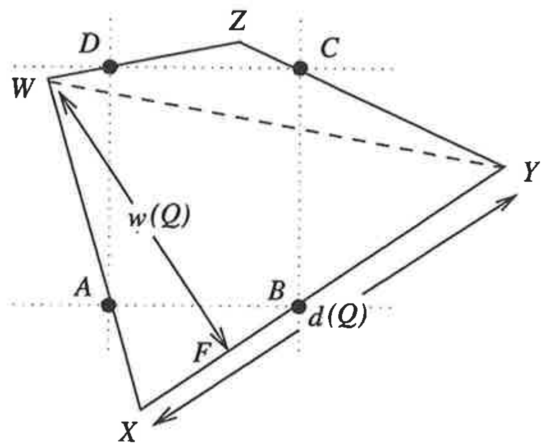


Figure 6.6: The quadrilateral Q with diameter XY

We first note that the diameter of a polygonal set is the maximal distance between a pair of vertices of the polygon. Suppose first that $d(Q)$ is the length of an edge, XY say, of Q. Without loss of generality, suppose that W is the vertex of Q furthest from b. Then $w(Q) \leq d(W, b)$. Let T be the triangle XYW. Clearly $d(T) = XY$ and so $w(T) = d(W, b)$. It follows that $w(Q) \leq w(T)$. But since $T \subseteq Q$, we have $w(T) \leq w(Q)$. Hence $w(Q) = w(T) = d(W, b)$. Since T and Q have the same width and diameter, it suffices to show that $f(T) < uv$. Noting that the edge WY contains no lattice points, T may be enlarged about the point X to $T_1 = \Delta W_1 X Y_1$ where $W_1 Y_1$ contains the point C or D, and C and D are both not in T_1^o. Now T_1 satisfies the conditions of the Triangle Rotation Lemma. By Lemma 6.4 there is a triangle Δ circumscribing a cell of Λ_R with $f(T_1) \leq f(\Delta)$. Furthermore, by Lemma 6.3, $f(\Delta) \leq uv$. It follows that

$f(K) \leq f(Q) = f(T) < f(T_1) \leq f(\triangle) \leq uv.$

We now suppose that $d(Q)$ is the length of a diagonal of Q, WY say. Let t be the width of Q in a direction perpendicular to WY (Figure 6.7). Since the (minimal) width of Q occurs in a direction perpendicular to an edge of Q, we have $w(Q) < t$. Let WY make an acute angle θ with CD and let XZ intersect WY in the point O. Now $A(Q) = \frac{1}{2}td(Q)$. This area is also obtained by adding the areas of the quadrilaterals $ODWA$, $OBYC$ to $OCZD$, $OAXB$.

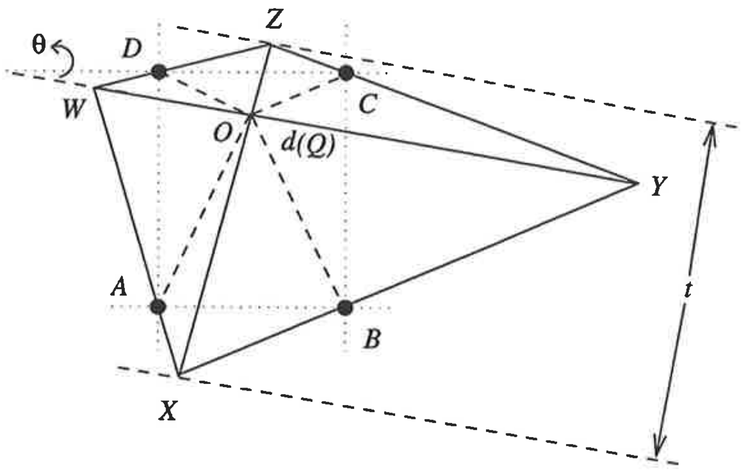


Figure 6.7: The quadrilateral Q with diameter WY

Suppose first that $AB = u$ and $BC = v$. Then we have

$$\frac{1}{2}td(Q) = \frac{1}{2}vd(Q)\cos\theta + \frac{1}{2}ut\cos\theta.$$

Hence

$$td(Q) = (tu + d(Q)v)\cos\theta \leq tu + d(Q)v.$$

Adding uv to both sides of the inequality and factorising, we have

$$(t - v)(d(Q) - u) \leq uv.$$

Since $w(Q) < t$, we have

$$f(Q) = (w(Q) - v)(d(Q) - u) < (t - v)(d(Q) - u) \leq uv.$$

Now suppose that $AB = v$ and $BC = u$. Repeating the above argument, we obtain the corresponding inequality

$$(w(Q) - u)(d(Q) - v) < uv.$$

By (6.5), $f(Q) = (w(Q) - v)(d(Q) - u) < uv$. □

6.3 Proof of Theorem 6.1

We now assume that K is a maximal set. By Lemma 6.3, $f(K) \geq uv$. Furthermore, for any convex set K, $(w - 2r)d \leq 2\sqrt{3}r^2$ (Scott 1979a).

We first suppose that $r \leq \frac{1}{2}u \leq \frac{1}{2}v$. Then

$$(w - v)(d - u) < (w - v)d \leq (w - 2r)d \leq 2\sqrt{3}r^2 \leq 2\sqrt{3}.\frac{u}{2}.\frac{v}{2} = \frac{\sqrt{3}}{2}uv < uv.$$

Hence K is not maximal. We may therefore assume that $r > \frac{1}{2}u$. Since we also have $w > v$, Lemma 5.4 applies and it follows that either $K \subseteq T$ where T is a triangle satisfying the conditions of the Triangle Rotation Lemma or $K \subseteq Q$ where Q is a proper convex quadrilateral circumscribing a cell of Λ_R with edges each containing in its interior a vertex of the cell. If $K \subseteq Q$, then $f(K) \leq f(Q)$. By Lemma 6.5, $f(K) < uv$ and so K is not maximal. Hence $K \subseteq T$. By Lemma 6.4, $f(K) \leq f(\triangle)$. Since K is maximal, it may be deduced that $K \cong \triangle$ where $\triangle$ is a triangle circumscribing a cell of Λ_R. By Lemma 6.3, $f(K) \leq uv$ with equality when and only when $K \cong \mathcal{T}_R$ (Figure 6.3). This completes the proof of Theorem 6.1.

6.4 Proof of Theorem 6.2

Let K now be a set with $G(K^o, \Gamma) = 2$. By $SL2$, we have the following two cases:

(i) If z_1 and z_2 are both odd, we consider the sublattice Γ' and observe that $G(K^o, \Gamma') = 0$. In this case $u = v = \sqrt{2}$ and by Theorem 6.1, we have

$$(w - \sqrt{2})(d - \sqrt{2}) \leq 2.$$

However,

$$\begin{aligned}(w-2)(d-1)-(w-\sqrt{2})(d-\sqrt{2}) &= w(\sqrt{2}-1)+d(\sqrt{2}-2)\\ &\le d(\sqrt{2}-1)+d(\sqrt{2}-2)\\ &= d(2\sqrt{2}-3)<0.\end{aligned}$$

It follows that $(w-2)(d-1)<(w-\sqrt{2})(d-\sqrt{2})\le 2$. Hence K is not maximal.

(ii) If say z_1 is odd and z_2 is even, we consider the sublattice Γ'' and observe that $G(K^o,\Gamma'')=0$. Here $u=1$, $v=2$ and by Theorem 6.1, we have

$$(w-2)(d-1)\le 2.$$

Equality occurs when and only when $K\cong\mathcal{T}_2$ (Figure 6.4).

6.5 Two corollaries

Corollary 6.6 *Let K be a set in $\mathcal{K}^2$ with $G(K^o,\Lambda_R)=0$. Then*

$$(w-v)d\le\frac{u}{2}(\sqrt{3}u+2v).$$

Equality is attained when and only when $K\cong\mathcal{E}_R$ (Figure 5.3).

Corollary 6.7 *Let K be a set in $\mathcal{K}^2$ with $G(K^o,\Gamma)=2$. Then*

$$(w-2)d\le 2+\frac{\sqrt{3}}{2}.$$

Equality is attained when and only when $K\cong\mathcal{E}_2$ (Figure 5.4).

To prove Corollary 6.6, we rearrange (6.3) to obtain

$$(w-v)d\le uw, \tag{6.6}$$

with equality when and only when $K\cong\mathcal{T}_R$ (Figure 6.3). Combining this with (5.3), we have

$$(w-v)d\le uw\le\frac{u}{2}(\sqrt{3}u+2v),$$

with equality when and only when $K \cong \mathcal{E}_R$ (Figure 5.3). Corollary 6.7 may be deduced in the same way by rearranging (6.4) to obtain $(w-2)d \leq w$ and combining this inequality with (5.4).

6.6 Related results

We observe that (6.1) may be rewritten as

$$\frac{1}{w} + \frac{1}{d} \geq 1. \tag{6.7}$$

McMullen and Wills (1981) give a generalization of (6.7) for sets in $\mathcal{K}^n$. The extensions are quite complicated and involve the use of functionals $w_i(K) = w_i$ and $\delta_i(K) = \delta_i$ defined as follows: Let $w_i(K) = w_i$ denote the width of K in a direction parallel to the ith basis vector (see Figure 6.8 for w_1 and w_2 in $\Re^2$). The functionals w_i are also called the *outer* 1-*quermasses* of K in the direction of the x_i-axis. Let H_i be a plane perpendicular to the ith basis vector. Then $\delta_i(K) = \delta_i$ is defined to be $\max w(K \cap H_i)$. We also recall the definition of $\triangle_n$ in Theorem 5.6. Then

Theorem 6.8 (McMullen and Wills) *Let $K \in \mathcal{K}^n$ and let $G(K^o, \Gamma_n) = 0$. Then*

$$\begin{aligned} \frac{\triangle_{n-1}}{\delta_i} + \frac{1}{w_i} &\geq 1, \quad i = 1, \ldots, n, \\ \sqrt{2}\frac{\triangle_{n-1}}{w} + \sqrt{n}\frac{1}{d} &\geq 1. \end{aligned}$$

In the same paper, the following stronger result is conjectured:

Conjecture 6.9 (McMullen and Wills) *Let $K \in \mathcal{K}^n$ and let $G(K^o, \Gamma_n) = 0$. Then*

$$\frac{\triangle_{n-1}}{w} + \frac{1}{d} \geq 1.$$

We note that in the case $n = 2$, we have (6.7).

Scott (1985b) and Wills (1990) give simpler generalizations of (6.7) by using the idea of an *axial diameter.* Let $d_i(K) = d_i$ denote the length of a maximal segment of K parallel to the ith basis vector (see Figure 6.8 for d_1 and d_2 in $\Re^2$). The functionals d_i are also called the *inner* 1-*quermasses* of K in the direction of the x_i-axis. In fact, in the case $n = 2$, we have $d_i = \delta_i$. Scott (1985b) proves the following analogue to (6.7):

Theorem 6.10 (Scott) *Let $K \in \mathcal{K}^n$ and let $G(K^o, \Gamma_n) = 0$. Then*

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq 1.$$

Equality holds when and only when K is a certain crosspolytope.

Wills (1990) improves Scott's result above by using the functionals w_i.

Theorem 6.11 (Wills) *Let $K \in \mathcal{K}^n$ and let $G(K^o, \Gamma_n) = 0. Then*

$$\frac{1}{w_i} + \sum_{j \neq i} \frac{1}{d_j} \geq 1, \quad i = 1, \ldots, n.$$

Equality holds when and only when K is a certain crosspolytope.

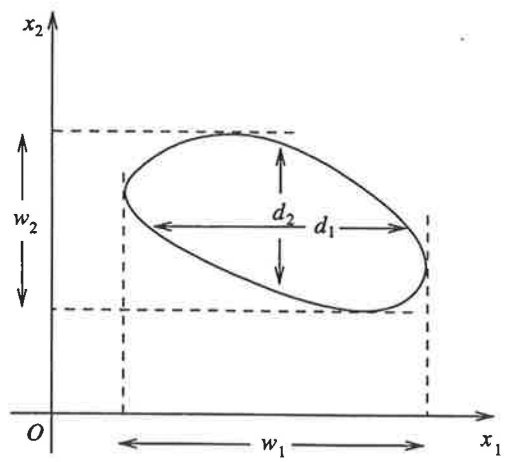


Figure 6.8: The functionals w_i and d_i for $\Re^2$

Recently, Vassallo and Wills (1996) generalized (6.7) to arbitrary lattices. Let λ_1 denote the length of the shortest non-zero vector of Λ and let $2\mu_1$ be the maximal distance between two adjacent lattice lines (we note here that $\lambda_1 = |\mathbf{a}|$ and $2\mu_1 = |\mathbf{b}|\sin\theta$, where $\{\mathbf{a}, \mathbf{b}\}$ is a Minkowski-reduced basis (p.64)). Then we have the following generalization of (6.7):

Theorem 6.12 (Vassallo and Wills) *Let $K \in \mathcal{K}^2$ and let $G(K^o, \Lambda) = 0$. Then*

$$\frac{2\mu_1}{w} + \frac{\lambda_1}{d} \geq 1.$$

Equality holds when and only when K is congruent to the triangle shown in Figure 6.9.

This result is further extended in the same paper to the case where K^o contains an arbitrary number of lattice points.

Theorem 6.13 (Vassallo and Wills) *Let $K \in \mathcal{K}^2$ and let $G(K^o, \Lambda) < k^2$. Then*

$$\frac{2\mu_1}{w} + \frac{\lambda_1}{d} \geq \frac{1}{k}.$$

By applying Theorem 6.12 to the rectangular lattice and rearranging the terms of the inequality, Theorem 6.1 follows immediately.

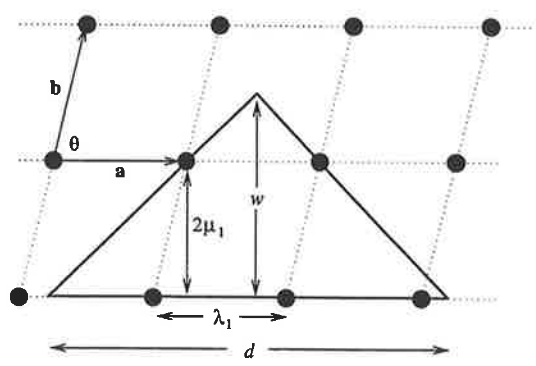


Figure 6.9: Vassallo's generalization of Scott's width-diameter inequality

Chapter 7

Further width inequalities for planar convex sets with lattice point constraints

7.1 Introduction

Let K be a set in $\mathcal{K}^2$ with area $A(K) = A$, perimeter $p(K) = p$, diameter $d(K) = d$ and circumradius $R(K) = R$. Scott (1980) proves that if $G(K^o, \Gamma) = 0$, then

$$
\begin{aligned}
(w-1)A &\leq \frac{1}{2}w^2, && (7.1)\\
(w-1)p &\leq 3w, && (7.2)\\
(w-1)R &\leq \frac{1}{\sqrt{3}}w. && (7.3)
\end{aligned}
$$

Equality is attained in (7.1) when and only when $K \cong \mathcal{T}_0$ (Figure 6.1) and equality is attained in (7.2) and (7.3) when and only when $K \cong \mathcal{E}_0$ (Figure 5.1).

In this chapter we generalize the above inequalities to rectangular lattices. We then use these results to obtain the analogue of (7.1) in the case where $G(K^o, \Gamma) = 1$. We also obtain the analogues of (7.1), (7.2) and (7.3) in the case where $G(K^o, \Gamma) = 2$. Let Λ_R denote the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$ where $u \leq v$. We prove

Theorem 7.1 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$
(w-v)A \leq \frac{1}{2}uw^2, \tag{7.4}
$$

$$(w-v)p \leq 3uw, \tag{7.5}$$

$$(w-v)R \leq \frac{1}{\sqrt{3}}uw. \tag{7.6}$$

Equality is attained in (7.4) *when and only when* $K \cong \mathcal{T}_R$ (Figure 6.3), *and equality is attained in* (7.5) *and* (7.6) *when and only when* $K \cong \mathcal{E}_R$ (Figure 5.3).

The next two results will quickly follow from Theorem 7.1.

Theorem 7.2 *Let K be a set in $\mathcal{K}^2$ with $G(K^o,\Gamma)=1$. Then*

$$(w-\sqrt{2})A \leq \frac{1}{\sqrt{2}}w^2. \tag{7.7}$$

Equality is attained when and only when $K \cong \mathcal{T}_1$ (Figure 6.2).

Theorem 7.3 *Let K be a set in $\mathcal{K}^2$ with $G(K^o,\Gamma)=2$. Then*

$$(w-2)A \leq \frac{1}{2}w^2, \tag{7.8}$$

$$(w-2)p \leq 3w, \tag{7.9}$$

$$(w-2)R \leq \frac{1}{\sqrt{3}}w. \tag{7.10}$$

Equality is attained in (7.8) *when and only when* $K \cong \mathcal{T}_2$ (Figure 6.4), *and equality is attained in* (7.9) *and* (7.10) *when and only when* $K \cong \mathcal{E}_2$ (Figure 5.4).

7.2 Proofs of (7.5) and (7.6) of Theorem 7.1

Suppose that $G(K^o,\Lambda_R)=0$. To prove (7.5) and (7.6) of Theorem 7.1, we adapt the method by Scott (1980). Hence to prove (7.6), we recall Jung's theorem which states that for a set $K \in \mathcal{K}^2$, $R \leq d/\sqrt{3}$, with equality when and only when K is an equilateral triangle (Yaglom and Boltyanskii 1961, p.17). By Jung's theorem and (6.6), we have

$$(w-v)R \leq (w-v)\frac{d}{\sqrt{3}} \leq \frac{1}{\sqrt{3}}uw.$$

For equality, we require $K \cong \mathcal{E}_R$.

We now prove (7.5). Let r be the inradius of K. We note first that if K is a convex polygon, K may be partitioned into triangles by joining each vertex of K to an in-centre of K. Summing the areas of these triangles gives

$$A \geq \frac{1}{2}pr, \tag{7.11}$$

with equality when and only when every edge of K touches the unique incircle. Since any set in $\mathcal{K}^2$ may be approximated by a convex polygon, this inequality is valid for all sets in $\mathcal{K}^2$. By using (7.11) and assuming (7.4) for the moment, we have

$$(w-v)p \leq \frac{2}{r}(w-v)A \leq \frac{uw^2}{r}.$$

By Blaschke's Theorem (Yaglom and Boltyanskii 1961, p.18), we have $w \leq 3r$, with equality when and only when K is an equilateral triangle. Hence

$$(w-v)p \leq 3uw,$$

with equality when and only when $K \cong \mathcal{E}_R$.

7.3 Proof of (7.4) of Theorem 7.1

We adapt the method by Scott (1980) to prove (7.4). We first observe that (7.4) may be written

$$\phi(K) = \frac{u}{2A} - \frac{w-v}{w^2} \geq 0.$$

The problem therefore becomes one of finding the minimal value of $\phi(K)$. Now

$$\begin{aligned} \frac{\partial\phi}{\delta A} &= -\frac{u}{2A^2} < 0, \\ \frac{\partial\phi}{\delta w} &= -\left(\frac{2v-w}{w^3}\right). \end{aligned}$$

Recalling Theorem 5.1 and noting that $u \leq v$, we have $w \leq v + \frac{1}{2}\sqrt{3}u < 2v$. It follows that $\partial\phi/\partial w < 0$ and hence ϕ is a decreasing function of A and w. Now let K be a set for which $\phi(K)$ is minimal. Since ϕ is a decreasing function of A and w, we choose K so that A and w are as large as possible.

We first note that $(w-2r)A \leq w^2r/\sqrt{3}$ (Scott 1979a). Hence if $r \leq \frac{1}{2}u \leq \frac{1}{2}v$, we have

$$(w-v)A \leq (w-2r)A \leq \frac{w^2r}{\sqrt{3}} \leq \frac{w^2}{\sqrt{3}}.\frac{u}{2} < \frac{1}{2}uw^2.$$

We may therefore assume that $r > \frac{1}{2}u$. Furthermore, since $w > v$, it follows from Lemma 5.4 that either $K \subseteq T$, where T is a triangle satisfying the conditions of the Triangle Rotation Lemma or $K \subseteq Q$ where Q is a proper convex quadrilateral circumscribing a cell of Λ_R, with edges each containing in its interior a vertex of the cell.

Suppose first that $K \subseteq Q$. Clearly, $\phi(K) \geq \phi(Q)$. Since $G(Q^o, \Lambda_R) = 0$, we may take $K = Q$. We now recall the following lemma by Scott (1993):

Lemma 7.4 (Scott) *The quadrilateral Q can be transformed into a kite Q' having the following properties:*

(a) $w(Q') \geq w(Q)$,

(b) *Q' contains no lattice point in its interior,*

(c) *the sides of Q' pass through the points* $(0,0)$, $(u,0)$, (u,v) *and* $(0,v)$,

(d) *Q' has its axis along* $x = \frac{1}{2}u$,

(e) $A(Q') \geq A(Q)$.

Property (e) is not explicitly stated in (Scott 1993) but follows from the fact that Q' is obtained from Q by Steiner symmetrization (which preserves areas) and an enlargement with scale factor $s \geq 1$. From Lemma 7.4 (a) and (e), it follows that $\phi(Q') \leq \phi(Q)$. We may therefore take K to be the kite $Q' = XYZW$ (Figure 7.1). We now show that $(w-v)A < \frac{1}{2}uw^2$.

Let $XZ = x$ and $YW = y$. Then $A = \frac{1}{2}xy$. Also, computing the area of Q'

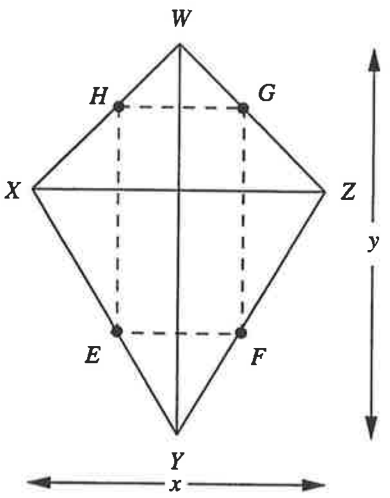


Figure 7.1: The kite Q'

in terms of the areas of the cell and the remaining triangles gives

$$A = uv + \frac{1}{2}v(x-u) + \frac{1}{2}u(y-v).$$

Hence $xy = vx + uy$.

Suppose first that $x \geq 2u$. Then $y \leq 2v$. From $xy = vx + uy$, we have $x = uy/(y-v)$ and hence

$$A = \frac{1}{2}\frac{uy^2}{(y-v)}.$$

We note that for the given range of y, $y^2/(y-v)$ is a decreasing function of y. Therefore since $w < y$, we have

$$A = \frac{1}{2}\frac{uy^2}{(y-v)} < \frac{1}{2}\frac{uw^2}{(w-v)},$$

that is,

$$(w-v)A < \frac{1}{2}uw^2.$$

Suppose now that $x < 2u$. Then rearranging $xy = vx + uy$, we have $y = vx/(x-u)$ and

$$A = \frac{1}{2}\frac{vx^2}{(x-u)}.$$

For the given range of x, $x^2/(x-u)$ is a decreasing function. Since $w < x$, we have

$$A = \frac{1}{2}\frac{vx^2}{(x-u)} < \frac{1}{2}\frac{vw^2}{(w-u)}.$$

Again, from Theorem 5.1, we have $w \le v + \frac{1}{2}\sqrt{3}u$. It follows that $w < u + v$. Hence $(v-u)w < (v^2 - u^2)$ or equivalently $v/(w-u) < u/(w-v)$. Hence

$$A < \frac{1}{2}\frac{vw^2}{(w-u)} < \frac{1}{2}\frac{uw^2}{(w-v)},$$

and we have

$$(w-v)A < \frac{1}{2}uw^2.$$

We now complete the proof by showing that if $K \subseteq T$, then $K \cong \mathcal{T}_R$ (Figure 6.3). Since $K \subseteq T$, we have $\phi(K) \ge \phi(T)$. By the Triangle Rotation Lemma, either T circumscribes a cell of Λ_R or there is a triangle $T_* \cong \lambda T$, $\lambda > 1$, with T_* circumscribing a cell of Λ_R. In either case, there is a triangle $\triangle$ circumscribing Λ_R with $A(\triangle) \ge A(T)$ and $w(\triangle) \ge w(T)$. Since ϕ is a decreasing function of A and w, we have $\phi(\triangle) \le \phi(T) \le \phi(K)$. Since $G(\triangle^o, \Lambda_R) = 0$, we may take $K \cong \triangle$. In this case, by Lemma 6.3, we have $(w-v)(d-u) \le uv$, with equality when and only when $K \cong \mathcal{T}_R$ (Figure 6.3). Rearranging the terms in the inequality, we have $d \le uw/(w-v)$. Hence

$$A = \frac{1}{2}wd \le \frac{1}{2}\frac{uw^2}{(w-v)},$$

that is,

$$(w-v)A \le \frac{1}{2}uw^2,$$

with equality when and only when $K \cong \mathcal{T}_R$ (Figure 6.3).

This completes the proof of (7.4) of Theorem 7.1

7.4 Proofs of Theorems 7.2 and 7.3

Let K now be a set with $G(K^o, \Gamma) = 1$. By $SL1$, we consider the sublattice Γ' and note that $G(K^o, \Gamma') = 0$. In this case $u = v = \sqrt{2}$ and by (7.4), we have

$$(w - \sqrt{2})A \leq \frac{1}{\sqrt{2}} w^2,$$

with equality when and only when $K \cong \mathcal{T}_1$ (Figure 6.2). Hence Theorem 7.2 is proved.

Now let $G(K^o, \Gamma) = 2$. By $SL2$, we have the following two cases:

(i) If z_1 and z_2 are both odd, we have $G(K^o, \Gamma') = 0$. Here again $u = v = \sqrt{2}$ and by Theorem 7.1, we have

$$\begin{aligned} (w - \sqrt{2})A &\leq \frac{\sqrt{2}}{2} w^2, \\ (w - \sqrt{2})p &\leq 3w\sqrt{2}, \\ (w - \sqrt{2})R &\leq \frac{\sqrt{2}}{\sqrt{3}} w. \end{aligned}$$

We also recall from Theorem 5.2 that $w \leq 2 + \frac{1}{2}\sqrt{3}$. Therefore

$$w < 2 + \sqrt{2} = \frac{\sqrt{2}}{\sqrt{2} - 1} = \frac{2\sqrt{2} - \sqrt{2}}{\sqrt{2} - 1}.$$

Rearranging terms, we have $\sqrt{2}(w - 2) < (w - \sqrt{2})$. Hence

$$\begin{aligned} (w - 2)A &< \frac{1}{2} w^2, \\ (w - 2)p &< 3w, \\ (w - 2)R &< \frac{1}{\sqrt{3}} w. \end{aligned}$$

(ii) If say z_1 is odd and z_2 is even, we consider the sublattice Γ'' and observe that $G(K^o, \Gamma'') = 0$. In this case $u = 1$ and $v = 2$ and we have Theorem 7.3. Equality is attained in (7.8) when and only when $K \cong \mathcal{T}_2$ (Figure 6.4) and equality is attained in (7.9) and (7.10) when and only when $K \cong \mathcal{E}_2$ (Figure 5.4).

7.5 Related results

We note that (7.4) has in fact been generalized to arbitrary lattices by Vassallo (1995). Vassallo makes use of the quantities λ_1 and $2\mu_1$ defined in §6.6. He proves

Theorem 7.5 (Vassallo) *Let $K \in \mathcal{K}^2$ and let $G(K^o, \Lambda) = 0$. Then*

$$(w - 2\mu_1)A \leq \frac{1}{2}\lambda_1 w^2.$$

Equality holds when and only when K is congruent to the triangle shown in Figure 6.9.

Although (7.4) follows immediately from Vassallo's result, the proof of (7.4) in §7.3 differs from Vassallo's proof in the use of the Triangle Rotation Lemma and Lemma 7.4 (Scott 1993), the proof of which depends on Steiner symmetrization.

Scott (1985c) gives a result relating V and the outer 1-quermasses, w_i (p.77) for sets in $\mathcal{K}^n$ having $G(K^o, \Gamma_n) = 0$.

Theorem 7.6 (Scott) *Let K be a set in $\mathcal{K}^n$ having $G(K^o, \Gamma_n) = 0$. Then*

(a) $V \leq \prod_{i=1}^{n} w_i$

(b) $V \leq \prod_{i=1}^{n} w_i - \frac{n^n}{n!}\prod_{i=1}^{n}(w_i - 1)$, *for* $1 \leq w_i \leq \frac{n}{n-1}$.

In the case where $n = 2$, $w_1 \geq 2$ and $w_2 > 1$, we have

(c) $V \leq w_1 w_2 - \frac{1}{2}w_1^2(w_2 - 1)$.

The bounds are best possible.

Chapter 8

Area-width relations for convex sets with lattice point constraints

8.1 Introduction

Let K be a non-empty set in $\mathcal{K}^2$ having area $A(K) = A$ and width $w(K) = w$. Let Λ_R be the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$, $u \leq v$. In Chapter 7, we obtained area-width inequalities for a set K having $G(K^o, \Lambda_R) = 0$. In this chapter we obtain a new area-width inequality for such a set. We then use this result to obtain the corresponding inequality for a set K having $G(K^o, \Gamma) = g$, where $g = 0, 2$. We also make a conjecture for the corresponding inequality for a set K having $G(K^o, \Gamma) = 1$.

Theorem 8.1 *Let K be a non-empty set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(v + \frac{\sqrt{3}}{2}u)^{-1}, \tag{8.1}$$

with equality when and only when $K \cong \mathcal{E}_R$ (Figure 5.3).

Corollary 8.2 *Let K be a non-empty set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 0$. Then*

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(1 + \frac{\sqrt{3}}{2})^{-1} \approx 0.309, \tag{8.2}$$

with equality when and only when $K \cong \mathcal{E}_0$ (Figure 5.1).

Corollary 8.3 *Let K be a non-empty set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 2$. Then*

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(2 + \frac{\sqrt{3}}{2})^{-1} \approx 0.201, \tag{8.3}$$

with equality when and only when $K \cong \mathcal{E}_2$ (Figure 5.4).

8.2 Proof of Theorem 8.1

The proof follows easily by combining two known results. The first is that of all sets in $\mathcal{K}^2$ with a given width, the equilateral triangle has the least area (Yaglom and Boltyanskii 1961, p.68). Hence $A \geq (1/\sqrt{3})w^2$. Combining this with (5.3), we have

$$\frac{A}{w^3} \geq (\frac{A}{w^2})\frac{1}{w} \geq \frac{1}{\sqrt{3}}(v + \frac{\sqrt{3}}{2}u)^{-1}.$$

Equality holds when and only when $K \cong \mathcal{E}_R$ (Figure 5.3).

Corollary 8.2 follows easily by letting $u = v = 1$ in Theorem 8.1. Equality holds when and only when $K \cong \mathcal{E}_0$ (Figure 5.1).

To prove Corollary 8.3, we use $SL2$ to obtain the following two cases:

(i) If z_1 and z_2 are both odd, we consider the sublattice Γ' and observe that $G(K^o, \Gamma') = 0$. Here $u = v = \sqrt{2}$ and by Theorem 8.1 we have

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(\sqrt{2} + \frac{\sqrt{3}}{2}\sqrt{2})^{-1} \approx 0.219 > \frac{1}{\sqrt{3}}(2 + \frac{\sqrt{3}}{2})^{-1} \approx 0.201.$$

(ii) If say, z_1 is odd and z_2 is even, we consider the sublattice Γ'' and observe that $G(K^o, \Gamma'') = 0$. Here $u = 1$ and $v = 2$ and by Theorem 8.1, we have

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(2 + \frac{\sqrt{3}}{2})^{-1} \approx 0.201.$$

Equality is attained when and only when $K \cong \mathcal{E}_2$ (Figure 5.4).

Hence Corollary 8.3 is proved.

8.3 A conjecture

We might now ask for a corresponding inequality for $G(K^o, \Gamma) = 1$. We make the following conjecture:

Conjecture 8.4 *Let K be a non-empty set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 1$. Then*

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}} \left(\frac{4}{\sqrt{2}(5+\sqrt{3})} \right) \approx 0.243,$$

with equality when and only when $K \cong \mathcal{E}_1$, where $\mathcal{E}_1$ is the equilateral triangle of Figure 8.1.

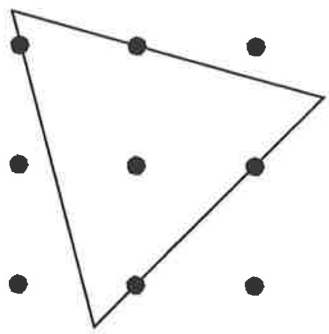


Figure 8.1: The equilateral triangle $\mathcal{E}_1$

The problem which occurs here is that for convex sets containing just one interior lattice point, $w \leq 1 + \sqrt{2} \approx 2.414$ with equality when and only when K is congruent to the isosceles triangle shown in Figure 5.2 (Scott 1985a). Since this set of largest width is not an equilateral triangle, the previous method cannot be applied.

A simple calculation shows that the width of $\mathcal{E}_1$ is $\frac{1}{4}\sqrt{2}(5+\sqrt{3}) \approx 2.38$. Hence if $0 < w \leq \frac{1}{4}\sqrt{2}(5+\sqrt{3})$, an equilateral triangle with interior containing one lattice point may be constructed. Since $A \geq (1/\sqrt{3})w^2$ with equality when and only when K is an equilateral triangle, for this range of w we have

$$\frac{A}{w^3} \geq (\frac{A}{w^2})\frac{1}{w} \geq \frac{1}{\sqrt{3}} \cdot \frac{4}{\sqrt{2}(5+\sqrt{3})} \approx 0.243.$$

Equality occurs when and only when $K \cong \mathcal{E}_1$.

This leaves unresolved those cases for which $\frac{1}{4}\sqrt{2}(5+\sqrt{3}) < w \leq 1 + \sqrt{2}$. We believe that the set for which A/w^3 is minimal is congruent to the equilateral triangle $\mathcal{E}_1$ of Figure 8.1.

Chapter 9

Circumradius-diameter and width-inradius inequalities for convex sets with and without lattice point constraints

9.1 Introduction

Let K be a set in $\mathcal{K}^2$ having diameter $d(K) = d$, width $w(K) = w$, circumradius $R(K) = R$ and inradius $r(K) = r$. Let Λ_R be the rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$, $u \leq v$. In this chapter, we establish an inequality relating R, d and w for a convex set without any lattice constraint. We then use this result to obtain an inequality relating d and R for a set K having $G(K^o, \Lambda_R) = 0$. By considering special rectangular lattices, we deduce the corresponding results for a set K having $G(K^o, \Gamma) = 0$ and $G(K^o, \Gamma) = 2$. We also obtain dual inequalities relating w and r for the lattice constrained sets. Finally we conjecture the corresponding results for the case where $G(K^o, \Gamma) = 1$.

Theorem 9.1 *Let K be a set in $\mathcal{K}^2$. Then*

$$2R - d \leq \frac{2}{3}(2 - \sqrt{3})w, \tag{9.1}$$

with equality when and only when K is an equilateral triangle.

Theorem 9.2 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$2R - d \leq \frac{2}{3}(2 - \sqrt{3})(v + \frac{\sqrt{3}}{2}u), \tag{9.2}$$

$$w - 2r \leq \frac{1}{3}(v + \frac{\sqrt{3}}{2}u), \tag{9.3}$$

with equality when and only when $K \cong \mathcal{E}_R$ (Figure 5.3).

Corollary 9.3 *Let* K *be a set in* $\mathcal{K}^2$ *with* $G(K^o, \Gamma) = 0$. *Then*

$$\begin{aligned} 2R - d &\leq \frac{1}{3}, \\ w - 2r &\leq \frac{1}{3}(1 + \frac{\sqrt{3}}{2}) \approx 0.622, \end{aligned}$$

with equality when and only when $K \cong \mathcal{E}_0$ (Figure 5.1).

Corollary 9.4 *Let* K *be a set in* $\mathcal{K}^2$ *with* $G(K^o, \Gamma) = 2$. *Then*

$$\begin{aligned} 2R - d &\leq \frac{1}{3}(5 - 2\sqrt{3}) \approx 0.512, \\ w - 2r &\leq \frac{1}{3}(2 + \frac{\sqrt{3}}{2}) \approx 0.955, \end{aligned}$$

with equality when and only when $K \cong \mathcal{E}_2$ (Figure 5.4).

9.2 Proof of Theorem 9.1

We may assume that $K^o \neq \emptyset$, for if $K^o = \emptyset$, then either $K = \emptyset$ or K is a line segment. If $K = \emptyset$, then Theorem 9.1 is trivially true. If K is a line segment then $d = 2R$, $w = 0$ and Theorem 9.1 is trivially true. Hence we may assume that $K \neq \emptyset$. It follows that $w \neq 0$. We now define the functional

$$f(K) = \frac{1}{w(K)}(2R(K) - d(K)) = \frac{1}{w}(2R - d).$$

We seek to maximize the functional $f(K)$. Clearly, since $d \leq 2R$, $f(K) \geq 0$. We first recall that the circumcircle of a set K either contains two diametrically opposite points of K or else it contains three points of ∂K which form the vertices of an acute-angled triangle (Yaglom and Boltyanskii 1961, p.59). In the first case, $2R = d$ and $f(K) = 0$, so K is not maximal. Hence we may assume that K contains an acute-angled triangle T with $R(T) = R(K)$. Furthermore since T is contained in K, $d(T) \leq d(K)$ and $w(T) \leq w(K)$. It follows that $f(K) \leq f(T)$. Hence it suffices to maximize $f(K)$ for acute-angled triangles T.

Let $T = \triangle XYZ$ be an acute-angled triangle with $\angle Y \leq \angle X \leq \angle Z$ as shown in Figure 9.1. Since $\angle Z$ is the largest angle, it follows that $XY = d$. We first apply to T a shear parallel to XY to obtain the triangle $T' = XYZ'$ with $YZ' = XY = d$. Let P and P' be the circumcentres of T and T' respectively. Since P and P' both lie on the perpendicular bisector of the line segment XY, and $PZ' > PZ = R(T)$, it follows that P' is further away from XY than the point P. Hence $R(T') > R(T)$. Furthermore $d(T') = d(T)$ and $w(T') = w(T)$. It follows that $f(T') \geq f(T)$. Hence we need only consider those cases for which T is an isosceles triangle with vertex angle at Y. In this case $\angle X = \angle Z = \alpha \geq \frac{\pi}{3}$. We note that $w = d \sin 2\alpha$ and that from the sine rule, $2R = d/\sin\alpha$. Hence we have

$$\begin{aligned} f(K) &= \frac{1}{w}(\frac{1}{\sin\alpha} - 1)d \\ &= (\frac{1}{\sin\alpha} - 1)(\frac{1}{\sin 2\alpha}). \end{aligned}$$

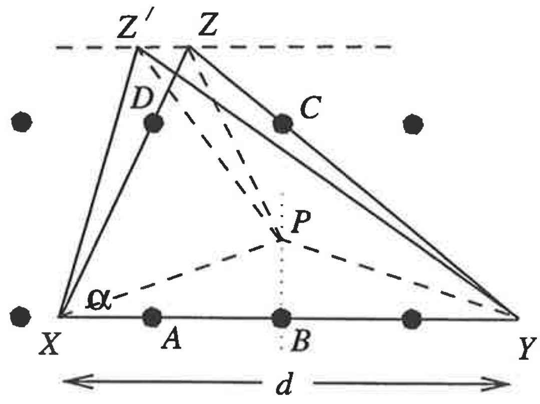


Figure 9.1: Shear applied to the triangle T

By letting $t = \tan\alpha$, we have

$$\begin{aligned} f(K) &= \left(\frac{\sqrt{1+t^2}}{t} - 1\right)\left(\frac{1+t^2}{2t}\right) \\ &= \frac{1}{2}(\sqrt{1+t^2} - t)\left(\frac{1+t^2}{t^2}\right) \\ &= \frac{1}{2}(\sqrt{1+t^2} - t)(\frac{1}{t^2} + 1) \\ &= \frac{1}{2}g(t)h(t). \end{aligned}$$

We note that

$$\begin{aligned} g(t) = \sqrt{1+t^2} - t > 0, \quad g'(t) &= \frac{t}{\sqrt{1+t^2}} - 1 < 0. \\ h(t) = \frac{1}{t^2} + 1 > 0, \quad h'(t) &= -\frac{2}{t^3} < 0. \end{aligned}$$

Since $f(K)$ is a product of positive, decreasing functions of t, it is itself a positive, decreasing function of t. Since $\alpha \geq \frac{\pi}{3}$, we have $t \geq \sqrt{3}$. Hence the maximal value of $f(K)$ is attained when $t = \sqrt{3}$, that is when T is an equilateral triangle. In this case

$$f(K) = \frac{1}{w}(2R - d) \leq \frac{2}{3}(2 - \sqrt{3}),$$

that is,

$$2R - d \leq \frac{2}{3}(2 - \sqrt{3})w,$$

with equality when and only when K is an equilateral triangle.

9.3 Proofs of Theorem 9.2 and corollaries

To prove Theorem 9.2, we combine (5.3) with (9.1) to obtain (9.2). Equality is attained when and only when $K \cong \mathcal{E}_R$ (Figure 5.3).

To prove (9.3), we recall the result of Blaschke, that for any set in $\mathcal{K}^2$, $w \leq 3r$ with equality when and only when K is an equilateral triangle (Yaglom and Boltyanskii 1961, p.18). This inequality may be rewritten as

$$w - 2r \leq \frac{w}{3}, \tag{9.4}$$

with equality when and only when K is an equilateral triangle. Combining this with (5.3), we obtain the required inequality.

Corollary 9.3 follows easily from Theorem 9.2 by letting $u = v = 1$.

We now let K be a set with $G(K^o, \Gamma) = 2$. By $SL2$ we have the following two cases:

(i) If z_1 and z_2 are both odd, then $G(K^o, \Gamma') = 0$. Here we have $u = v = \sqrt{2}$ and by Theorem 9.2,

$$\begin{aligned} 2R - d &\leq \frac{1}{3}\sqrt{2} \approx 0.4714 < \frac{1}{3}(5 - 2\sqrt{3}) \approx 0.512, \\ w - 2r &\leq \frac{\sqrt{2}}{3}(1 + \frac{\sqrt{3}}{2}) \approx 0.879 < \frac{1}{3}(2 + \frac{\sqrt{3}}{2}) \approx 0.955. \end{aligned}$$

(ii) If say, z_1 is odd and z_2 is even, we consider the sublattice Γ'' and observe that $G(K^o, \Gamma'') = 0$. In this case $u = 1$ and $v = 2$ and by Theorem 9.2, we have

$$\begin{aligned} 2R - d &\leq \frac{1}{3}(5 - 2\sqrt{3}) \approx 0.512, \\ w - 2r &\leq \frac{1}{3}(2 + \frac{\sqrt{3}}{2}) \approx 0.955. \end{aligned}$$

Equality is attained when and only when $K \cong \mathcal{E}_2$ (Figure 5.4).

Hence Corollary 9.4 is proved.

9.4 A conjecture and related results

As in §8.3, we now make a conjecture for the corresponding inequalities for a set K having $G(K^o, \Gamma) = 1$.

Conjecture 9.5 *Let K be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 1$. Then*

$$\begin{aligned} 2R - d &\leq \sqrt{2}(\frac{7}{6} - \frac{\sqrt{3}}{2}) \approx 0.425, \\ w - 2r &\leq \frac{\sqrt{2}}{12}(5 + \sqrt{3}) \approx 0.793, \end{aligned}$$

with equality when and only when $K \cong \mathcal{E}_1$ (Figure 8.1).

As in §8.3, the difficulty which occurs here is that for convex sets containing just one interior lattice point, $w \leq 1 + \sqrt{2}$ with equality when and only when K is congruent to the isosceles triangle shown in Figure 5.2 (Scott 1985a). As this set of largest width is not an equilateral triangle, Theorem 9.1 does not give a sharp inequality.

Using the same arguments given in §8.3, we first consider the case where $0 < w \leq \frac{1}{4}\sqrt{2}(5+\sqrt{3})$. In this case, an equilateral triangle with interior containing one lattice point may be constructed. It follows from (9.1) and (9.4) that for this given range of w,

$$\begin{aligned} 2R - d &\leq \sqrt{2}(\frac{7}{6} - \frac{\sqrt{3}}{2}) \approx 0.425, \\ w - 2r &\leq \frac{\sqrt{2}}{12}(5 + \sqrt{3}) \approx 0.793, \end{aligned}$$

with equality when and only when $K \cong \mathcal{E}_1$ (Figure 8.1).

As in §8.3, this leaves unresolved those cases for which $\frac{1}{4}\sqrt{2}(5 + \sqrt{3}) < w \leq 1 + \sqrt{2}$. We believe that the set for which $(2R - d)$ and $(w - 2r)$ are maximal is congruent to the equilateral triangle $\mathcal{E}_1$ (Figure 8.1).

Scott (1978b, 1979a, 1981) has obtained a number of inequalities involving the quantities $(w - 2r)$ and $(2R - d)$ for sets without lattice constraints in $\mathcal{K}^2$. We summarize the results concerning $(w - 2r)$ and $(2R - d)$ in Theorem 9.6 and Theorem 9.7 respectively. The set given in brackets is the best possible set.

Theorem 9.6 (Scott) *Let K be a set in $\mathcal{K}^2$. Then*

(i) $(w - 2r)d \leq 2\sqrt{3}r^2$ *(Equilateral triangle)*

(ii) $(w - 2r)d < w^2/2$ *(Infinite isosceles triangle)*

(iii) $(w - 2r)d \leq 2wr/\sqrt{3}$ *(Equilateral triangle)*

(iv) $(w - 2r)R < w^2/4$ *(Infinite isosceles triangle)*

(v) $(w - 2r)R \leq 2r^2$ *(Equilateral triangle)*

(vi) $(w - 2r)R \leq 2wr/3$ *(Equilateral triangle)*

(vii) $(w - 2r)p \leq 2w^2/\sqrt{3}$ *(Equilateral triangle)*

(viii) $(w - 2r)p \leq 2\sqrt{3}wr \leq 6\sqrt{3}r^2$ *(Equilateral triangle)*

(ix) $(w - 2r)A < w^3/4$ *(Infinite isosceles triangle)*

(x) $(w - 2r)A \leq w^2r/\sqrt{3}$ *(Equilateral triangle)*

(xi) $(w-2r)A \le \sqrt{3}wr^2 \le 3\sqrt{3}r^3$ (*Equilateral triangle*).

Theorem 9.7 (Scott) *Let K be a set in $\mathcal{K}^n$. Then*

(i) *For $n = 2$, $(2R-d)A \le \pi(3\sqrt{3}-5)R^3$ (not best possible)*

(ii) *For $n = 2$, $(2R-d)p \le (2\sqrt{3}-3)\pi R^2$ (Sets of constant width)*

(iii) $(2R-d)d \le (2\sqrt{n+1}/n)(\sqrt{2n}-\sqrt{n+1})R^2$ *(Regular simplex)*

(iv) $(2R-d)w \le (2\sqrt{n+1}/n)(\sqrt{2n}-\sqrt{n+1})R^2$ *(Sets of constant width containing a regular simplex of width w)*

(v) $(2R-d)r \le (\sqrt{2}/n)(3\sqrt{n(n+1)}-\sqrt{2}(2n+1))R^2$ *(Sets of constant width containing a regular simplex of width w)*.

Chapter 10

New inequalities concerning the inradius of a lattice constrained convex set

10.1 Introduction

Let K be a set in $\mathcal{K}^2$ with area $A(K) = A$, perimeter $p(K) = p$, diameter $d(K) = d$ and inradius $r(K) = r$. In this chapter we obtain new inequalities for the pairs (A, r), (p, r) and (d, r) for a set K having $G(K^o, \Gamma) = 0$ (Awyong and Scott 1996b). By considering a special sublattice of the integral lattice, we also obtain an inequality concerning A and r for a set K having $G(K^o, \Gamma) = 1$. We prove

Theorem 10.1 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 0$. Then*

$$(2r - 1)A \leq 2(\sqrt{2} - 1) \approx 0.828, \tag{10.1}$$

with equality when and only when $K \cong \mathcal{S}_0$ (Figure 10.1).

strange! (that's not true in a strong sense!)

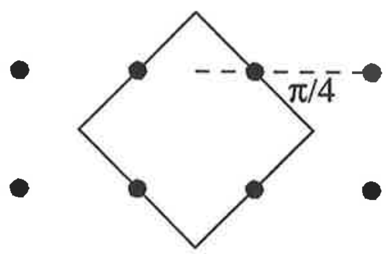


Figure 10.1: The diagonal square $\mathcal{S}_0$

Corollary 10.2 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 1$. Then*

$$(2r - \sqrt{2})A \leq 4(2 - \sqrt{2}) \approx 2.343, \tag{10.2}$$

with equality when and only when $K \cong \mathcal{S}_1$ (Figure 10.2).

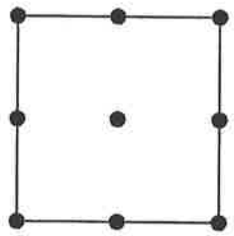


Figure 10.2: The square $\mathcal{S}_1$

Theorem 10.3 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 0$. Then*

$$(2r - 1)|A - 1| < \frac{1}{2}, \tag{10.3}$$

$$(2r - 1)|p - 4| < 2, \tag{10.4}$$

$$(2r - 1)(d - 1) < 1. \tag{10.5}$$

The limiting infinite strip $\mathcal{P}_0$ (Figure 10.3) *shows that the stated bounds are best possible.*

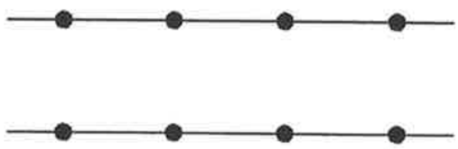


Figure 10.3: The infinite strip $\mathcal{P}_0$

10.2 Proofs of Theorem 10.1 and Corollary 10.2

We first prove a useful lemma.

Lemma 10.4 *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 0$. Then there is a set K_* in $\mathcal{K}^2$ with $G(K_*^o, \Gamma) = 0$ satisfying the following conditions:*
(a) $A(K_*) = A(K)$, $r(K_*) \geq r(K)$
(b) K_* *is symmetric about the lines* $x = \frac{1}{2}$, $y = \frac{1}{2}$.

Proof. We use Steiner symmetrization to obtain the set K_*. We first symmetrize K with respect to the line $x = \frac{1}{2}$ to obtain the set K_1. From Theorem 2.3, we know that $K_1 \in \mathcal{K}^2$, $A(K_1) = A(K)$ and $r(K_1) \geq r(K)$.

We now show that $G(K_1^o, \Gamma) = 0$. Since $G(K^o, \Gamma) = 0$, K^o intersects the line $y = k$, where k is an integer, either in the empty set or in a line segment of length at most 1. Hence the symmetric set K_1^o intersects the line $y = k$ either in the empty set or between the points $(0, k)$ and $(1, k)$. Clearly, $G(K_1^o, \Gamma) = 0$.

We now symmetrize K_1 with respect to the line $y = \frac{1}{2}$ to obtain K_*. By Theorem 2.3 again, we have $A(K_*) = A(K_1)$, $r(K_*) \geq r(K_1)$. It may also be deduced as above that $G(K_*^o, \Gamma) = 0$. Hence $A(K_*) = A(K)$ and $r(K_*) \geq r(K)$. By construction, K_* is symmetric about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ and the lemma is proved. $\square$

Let $f(K) = (2r(K) - 1)A(K) = (2r - 1)A$. By Lemma 10.4 we have $f(K) \leq f(K_*)$. It therefore suffices to prove Theorem 10.1 for sets K which are symmetric about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

To fully utilize the symmetry of K about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, we move the origin to the point $(\frac{1}{2}, \frac{1}{2})$. If $r \leq \frac{1}{2}$, (10.1) is trivially true. Hence we may assume that $r > \frac{1}{2}$. Since K^o does not contain the points $P_1(\frac{1}{2}, \frac{1}{2})$, $P_2(-\frac{1}{2}, \frac{1}{2})$, $P_3(-\frac{1}{2}, -\frac{1}{2})$ and $P_4(\frac{1}{2}, -\frac{1}{2})$, it follows by the convexity of K that for each $i = 1, \ldots, 4$, K is bounded by a line l_i through the point P_i, with l_1 and l_3 having negative slope and l_2 and l_4 having positive slope. Furthermore since K is symmetric about the coordinate axes, K is contained in a rhombus Q determined by the lines l_i, $i = 1, \ldots, 4$. Since $K \subseteq Q$, $A(K) \leq A(Q)$, $r(K) \leq r(Q)$ and we have $f(K) \leq f(Q)$. It is therefore sufficient to maximize $f(K)$ over the set of all rhombi, $K = Q$, determined by the lines l_i, $i = 1, \ldots, 4$ (Figure 10.4).

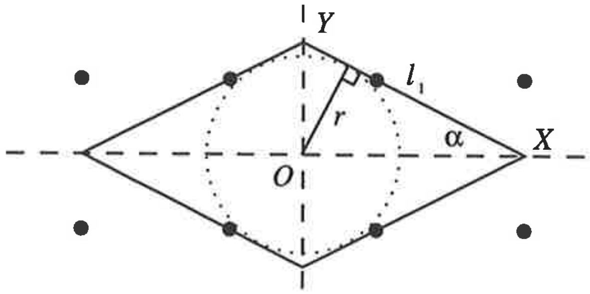


Figure 10.4: The rhombus Q

Let the line l_1 make an acute angle of α with the x-axis and let it intercept the x and y axes in the points $X(x, 0)$ and $Y(0, y)$ respectively. Since l_1 passes through $(\frac{1}{2}, \frac{1}{2})$, similar triangles give

$$\frac{y}{x} = \frac{\frac{1}{2}}{x - \frac{1}{2}},$$

that is,

$$\frac{1}{x} + \frac{1}{y} = 2.$$

Multiplying both sides of the equation by r, we get

$$2r = \frac{r}{x} + \frac{r}{y} = \sin\alpha + \cos\alpha.$$

Now

$$\begin{aligned} A &= 4.A(\Delta OXY) \\ &= 2xy \\ &= \frac{2r^2}{\sin\alpha\cos\alpha} \\ &= \frac{4r^2}{(\sin\alpha + \cos\alpha)^2 - 1} \\ &= \frac{4r^2}{4r^2 - 1} \\ &= 1 + \frac{1}{4r^2 - 1}. \end{aligned}$$

Hence

$$f(K) \;=\; (2r-1)A = 2r - 1 + \frac{1}{2r+1} = g(r). \qquad (10.6)$$

Now $\frac{1}{2}g'(r) = 1 - 1/(2r+1)^2 > 0$. Hence g is an increasing function of r. Noting that $\frac{1}{2} < r \le \frac{1}{2}\sqrt{2}$, the maximal value of g is therefore attained at $r = \frac{1}{2}\sqrt{2}$, that is, when and only when $K \cong \mathcal{S}_0$. In this case,

(not quite correct for the equality case!)

$$f(K) \le 2(\sqrt{2}-1) \approx 0.828.$$

We next use Theorem 10.1 to prove Corollary 10.2. Let K now be a convex set with $G(K^o, \Gamma) = 1$. By $SL1$, we consider the sublattice Γ' and note that $G(K^o, \Gamma') = 0$. Hence letting A' and r' be the area and the inradius respectively of K measured in the scale of Γ', and applying (10.1) to K with respect to Γ', we have

!??

$$(2r'-1)A' \le 2(\sqrt{2}-1),$$

with equality when and only when $K \cong \mathcal{S}_1$. Since Γ' is a rotation of Γ scaled by a factor of $\sqrt{2}$,

$$A' = (\frac{1}{\sqrt{2}})^2 A, \qquad r' = \frac{1}{\sqrt{2}} r,$$

where A and r are the area and the inradius respectively of K measured in the scale of the integral lattice Γ. Hence

$$(2.\frac{1}{\sqrt{2}}r - 1)\frac{A}{2} \le 2(\sqrt{2}-1).$$

Simplifying, we get

$$(2r - \sqrt{2})A \le 4(2-\sqrt{2}) \approx 2.343,$$

with equality when and only when $K \cong \mathcal{S}_1$.

10.3 Proof of Theorem 10.3

We first note that if $r \le \frac{1}{2}$, inequalities (10.3) and (10.4) are trivially true. Hence we need only consider those cases for which $\frac{1}{2} < r \le \frac{1}{2}\sqrt{2}$.

To prove (10.3), we first consider $A \le 1$. Since $r > \frac{1}{2}$, we have $A > \frac{1}{4}\pi$ and so

$$(2r-1)|A-1| = (2r-1)(1-A) < (\sqrt{2}-1)(1-\frac{\pi}{4}) < \frac{1}{2}.$$

Hence we may assume that $A > 1$. Using the same arguments as those given in §10.2, it suffices to consider a set K where K is a rhombus of the type described in Figure 10.4. Let $Q(r)$ denote such a rhombus with inradius r. From (10.6) we have

$$(2r-1)|A-1| = (2r-1)(A-1) = (2r-1)A - (2r-1) = \frac{1}{2r+1} < \frac{1}{2}.$$

Taking the infinite strip to be the limit of $Q(r)$ as r tends to $\frac{1}{2}$, it is seen that the stated bound is best possible.

To prove (10.4), we first consider $p \le 4$. Since $r > \frac{1}{2}$, we have $p > \pi$ and so

$$(2r-1)|p-4| = (2r-1)(4-p) < (\sqrt{2}-1)(4-\pi) < 2.$$

Hence we may assume that $p > 4$. We recall that $A \ge \frac{1}{2}pr$ (see inequality (7.11)). Combining this inequality with (10.3) and noting that $r > \frac{1}{2}$, we have

$$(2r-1)|p-4| = (2r-1)(p-4) \le (2r-1)(\frac{2A}{r}-4) \le 4(2r-1)(A-1) \le 4.\frac{1}{2} = 2,$$

obtaining (10.4). As before, taking the infinite strip to be the limit of $Q(r)$ as r tends to $\frac{1}{2}$, the stated bound is best possible.

Finally, to prove (10.5), we note from (6.1) that $(w-1)(d-1) \le 1$ with equality when and only when $K \cong T_0$ (Figure 6.1). Since $w \ge 2r$, we have

$$(2r-1)(d-1) \le (w-1)(d-1) \le 1.$$

Taking the infinite strip to be the limit of a sequence of triangles of the type T_0 shown in Figure 6.1 as w tends to $2r$, it can be seen that the stated bound is best possible.

10.4 Comment

Inequality (10.5) may be generalized to rectangular lattices by noting from (6.3) that $(w-v)(d-u) \leq uv$, with equality when and only when $K \cong \mathcal{T}_R$ (Figure 6.3). Since $w \geq 2r$, we have

$$(2r - v)(d - u) \leq (w - v)(d - u) \leq uv.$$

Taking the infinite strip to be the limit of a sequence of triangles of the type T_R shown in Figure 6.3 as w tends to $2r$, it follows that

$$(2r - v)(d - u) < uv.$$

Chapter 11

Area-diameter relations for convex sets containing one or two lattice points

11.1 Introduction

Let K be a non-empty set in $\mathcal{K}^2$ with area $A(K) = A$, diameter $d(K) = d$ and circumradius $R(K) = R$. Scott (1974a) proves

Theorem 11.1 (Scott) *Let K be a non-empty set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 0$. Then*

$$\frac{A}{d} \leq \lambda \approx 1.144, \tag{11.1}$$

where $\lambda = 2\sqrt{2}\sin\frac{\phi}{2}$, ϕ being the unique solution of the equation $\sin\theta = \frac{\pi}{2} - \theta$ ($\phi \approx 0.832 \approx 47.7^o$). The result is best possible with equality when and only when $K \cong \mathcal{H}_0$ (Figure 11.1).

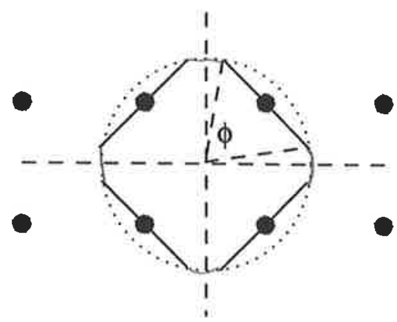


Figure 11.1: The truncated diagonal square $\mathcal{H}_0$, $\phi \approx 47.7^o$

Using $d \leq 2R$ and (11.1), it may be easily deduced that

$$\frac{A}{R} \leq 2\lambda \approx 2.288, \tag{11.2}$$

with equality when and only when $K \cong \mathcal{H}_0$.

Hammer (1979) proves

Theorem 11.2 (Hammer) *Let K be a non-empty set in $\mathcal{K}^2$ and suppose that $A/d > r\lambda$. Then $G(K^o, \Gamma) \geq r^2$.*

From Theorem 11.2 it may be deduced that if $G(K^o, \Gamma) < 2^2 = 4$, then $A/d \leq 2\lambda \approx 2.288$. In this chapter we establish sharp inequalities for A/d and A/R for a set K having $G(K^o, \Gamma) = 1$. We prove the following:

Theorem 11.3 *Let K be a non-empty set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 1$. Then*

$$\frac{A}{d} \leq \sqrt{2}\lambda \approx 1.616, \tag{11.3}$$

$$\frac{A}{R} \leq 2\sqrt{2}\lambda \approx 3.232, \tag{11.4}$$

where $\lambda = 2\sqrt{2}\sin\frac{\phi}{2}$, ϕ being the unique solution of the equation $\sin\theta = \frac{\pi}{2} - \theta$ ($\phi \approx 0.832 \approx 47.7^o$). Equality holds when and only when $K \cong \mathcal{H}_1$ (Figure 11.2).

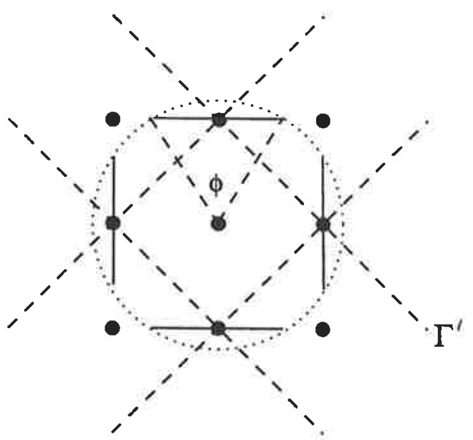


Figure 11.2: The truncated square $\mathcal{H}_1$, $\phi \approx 47.7^o$

Let $\mu = \sqrt{2}\lambda$. We give an analogue of Theorem 11.2.

Corollary 11.4 *Let K be a non-empty set in $\mathcal{K}^2$ and suppose that $A/d > r\mu$. Then $G(K^o, \Gamma) \geq 2r^2$.*

We also obtain the corresponding inequalities for a special class of sets K having $G(K^o, \Gamma) = 2$.

Theorem 11.5 *Let K be a symmetric, non-empty set in $\mathcal{K}^2$ containing two interior lattice points symmetrically placed about the centre of K. Then*

$$\frac{A}{d} \leq \alpha \approx 1.841 \tag{11.5}$$

$$\frac{A}{R} \leq 2\alpha \approx 3.682. \tag{11.6}$$

Equality holds when and only when $K \cong \mathcal{H}_2$ (Figure 11.3).

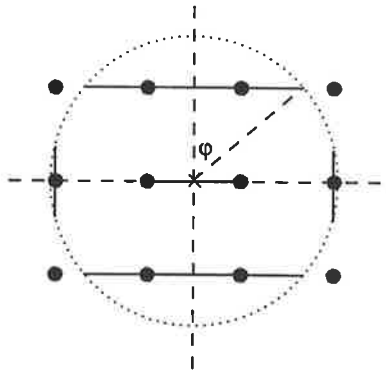


Figure 11.3: The truncated rectangle $\mathcal{H}_2$, $\varphi \approx 48.5^o$

In §11.2 and §11.3, we prove Theorem 11.3 and Corollary 11.4 respectively. In §11.4, we simplify the two lattice point problem for a special class of sets satisfying the conditions of Theorem 11.5. In §11.5 we prove Theorem 11.5 for this special class of sets. In §11.6, we complete the proof of Theorem 11.5. Finally, in §11.7 we conjecture a stronger form of Theorem 11.5.

11.2 Proof of Theorem 11.3

Suppose that $G(K^o, \Gamma) = 1$. By $SL1$, we consider the sublattice Γ' and note that $G(K^o, \Gamma') = 0$. Let A' and d' denote the area and the diameter respectively of K

measured in the scale of Γ'. Observing that Γ' is essentially a rotated version of Γ scaled by a factor of $\sqrt{2}$, we have

$$A' = (\frac{1}{\sqrt{2}})^2 A, \quad d' = \frac{1}{\sqrt{2}} d.$$

Furthermore from (11.1), we have $A'/d' \leq \lambda$. It follows that

$$\frac{A}{d} = \sqrt{2}\frac{A'}{d'} \leq \sqrt{2}\lambda \approx 1.616.$$

Equality holds when and only when $K \cong \mathcal{H}_1$ (Figure 11.2).

Since $d \leq 2R$, we have

$$\frac{A}{R} \leq 2\frac{A}{d} \leq 2\sqrt{2}\lambda,$$

with equality when and only when $K \cong \mathcal{H}_1$.

11.3 Proof of Corollary 11.4

We reproduce Hammer's proof with minor modifications. Let K be a non-empty set in $\mathcal{K}^2$ satisfying $A/d > r\mu$ where $\mu = \sqrt{2}\lambda$. If $r = 0$, the result of Corollary 11.4 is obvious. If $r = 1$, by Theorem 11.3, $G(K^o, \Gamma) \geq 2$. Hence we may assume that $r \geq 2$. Following Hammer (1979), we consider the similarity transformation

$$K \to K_1 = \frac{1}{r}K = \{\frac{1}{r}k : k \in K\}.$$

Therefore $A(K_1) = A(K)/r^2$ and $d(K_1) = d(K)/r$. Let

$$\mathcal{L} = \{(a, b) \in \Gamma; 0 \leq a, b \leq r - 1\}.$$

For a particular choice of $L \in \mathcal{L}$, we consider the translation

$$K_1 \to K_2 = K_1 - \frac{1}{r}L = \{k_1 - \frac{1}{r}L : k_1 \in K_1\}.$$

Since areas and diameters are unaltered by translations, we have

$$\frac{A(K_2)}{d(K_2)} = \frac{A(K_1)}{d(K_1)} = \frac{1}{r}\frac{A(K)}{d(K)} > \mu.$$

From Theorem 11.3 we know that K_2^o contains at least two distinct lattice points G_1 and G_2. Therefore K_1^o contains the two distinct points $G_1 + (1/r)L$ and $G_2 + (1/r)L$. This implies that K^o contains the two distinct lattice points $rG_1 + L$ and $rG_2 + L$. Let $X_1 = \{rG_1 + L; L \in \mathcal{L}\}$ and $X_2 = \{rG_2 + L; L \in \mathcal{L}\}$. Now $X_1 \cup X_2 \subset K^o$. We use the notation $|X|$ to denote the number of elements in a finite set X. Since L may be chosen in r^2 ways, $|X_1| = |X_2| = r^2$. Furthermore $X_1 \cap X_2 = \emptyset$, since if $rG_1 + (a,b) = rG_2 + (c,d)$ where $0 \le a,b,c,d \le r-1$, then $r(G_1 - G_2) = (c-a, d-b)$. Since r does not divide $c-a$ and $d-b$, we have $G_1 = G_2$. Hence $|X_1 \cup X_2| = 2r^2$ and K^o contains at least $2r^2$ lattice points.

We will now give a condition under which Corollary 11.4 improves Theorem 11.2. Let K be a set in $\mathcal{K}^2$ for which $A/d > x$ and let $[a]$ denote the greatest integer less than or equal to a. Then

$$\frac{A}{d} > x = \frac{x}{\lambda}.\lambda \ge [\frac{x}{\lambda}]\lambda.$$

By Theorem 11.2, $G(K^o, \Gamma) \ge [x/\lambda]^2$. We also have

$$\frac{A}{d} > x = \frac{x}{\mu}.\mu \ge [\frac{x}{\mu}]\mu,$$

where $\mu = \sqrt{2}\lambda$. By Corollary 11.4, $G(K^o, \Gamma) \ge 2[x/\mu]^2$. We will now show that if x satisfies $k\sqrt{2}\lambda \le x < ([k\sqrt{2}] + 1)\lambda$ for some $k \in \mathbf{Z}$, then $2[x/\mu]^2 > [x/\lambda]^2$. In other words, Corollary 11.4 improves Theorem 11.2. Since $[k\sqrt{2}]\lambda < k\sqrt{2}\lambda$, we have $[k\sqrt{2}]\lambda < x < ([k\sqrt{2}] + 1)\lambda$. Hence $[k\sqrt{2}] < x/\lambda < ([k\sqrt{2}] + 1)$, which implies that $[x/\lambda] = [k\sqrt{2}]$. By Theorem 11.2, $G(K^o, \Gamma) \ge [k\sqrt{2}]^2$. On the other hand, we also have $([k\sqrt{2}] + 1)\lambda < (k+1)\sqrt{2}\lambda$, so that $k\sqrt{2}\lambda \le x < (k+1)\sqrt{2}\lambda$. Thus $k\mu \le x < (k+1)\mu$. This implies that $k \le x/\mu < k+1$. Hence $[x/\mu] = k$. By Corollary 11.4, $G(K^o, \Gamma) \ge 2k^2$. Clearly $2k^2 > [k\sqrt{2}]^2$ and so Corollary 11.4 improves Theorem 11.2 if $k\sqrt{2}\lambda \le x < ([k\sqrt{2}] + 1)\lambda$ for some $k \in \mathbf{Z}$.

11.4 Some preliminaries

Let K now be a non-empty set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 2$. Without losing generality, we may assume that the origin O is one of the lattice points. Let L denote the other lattice point contained in K^o and let the coordinates of L be (z_1, z_2). From §2.5, we may assume that $z_1 \geq z_2$ and z_1 and z_2 are relatively prime. We will consider the following cases for L:

(i) $L(1,0)$ (ii) $L(1,1)$ (iii) $L(z_1, z_2), z_2 \geq 1$.

In the rest of this section and in §11.5, we will consider case (i). Cases (ii) and (iii) are dealt with in §11.6.

Let L be the point with coordinates $(1,0)$. Hence, the centre, M, of K is the point $M(\frac{1}{2}, 0)$. We now employ Steiner symmetrization to transform the set K into a set K_* which is symmetric about the lines $x = \frac{1}{2}$ and $y = 0$. We recall from Theorem 2.3 that Steiner symmetrization preserves convexity and areas and does not increase diameters. Hence

$$\frac{A(K)}{d(K)} \leq \frac{A(K_*)}{d(K_*)}.$$

To obtain K_*, we symmetrize K with respect to the line $x = \frac{1}{2}$ to obtain a closed convex set K_*. Since K is symmetric about M, K_* is also symmetric about $y = 0$. Furthermore since K^o does not contain any lattice points on the lines $y = \pm 1$, it intersects these lines in open segments of length at most 1. Hence K_*^o intersects the lines $y = \pm 1$ in the empty set or in line segments of lengths at most one unit. It follows that K_*^o does not contain the points $(1, \pm 1)$ and $(0, \pm 1)$. We now note that since $(2,0) \notin K^o$ and since K is convex, K is bounded by a line l_1 containing the point $(2,0)$. Since K is symmetric about $M(\frac{1}{2}, 0)$, K is also bounded by a line l_2 containing the point $(-1,0)$ and parallel to l_1. Therefore K lies in a strip bounded by parallel lines l_1 and l_2 containing the points $(2,0)$ and $(-1,0)$ respectively. Hence K_* is bounded by the lines $x = 2$ and $x = -1$ (Figure 11.4).

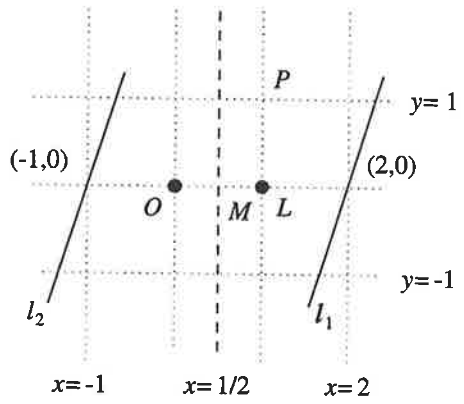


Figure 11.4: The case $L(1,0)$

To fully utilize the symmetry of K_*, we will translate the origin to the point $M(\frac{1}{2},0)$. The set K_* is now symmetric about the coordinate axes and is bounded by the lines $x = \pm\frac{3}{2}$. Furthermore K_* does not contain the points $(\pm\frac{1}{2},\pm 1)$. In particular, K_* does not contain the point $P(\frac{1}{2},1)$. Hence K_* is bounded by a line l containing the point P and having a non-positive slope m.

Henceforth we shall use A and d to denote $A(K_*)$ and $d(K_*)$ respectively. Since K_* is symmetric about the coordinate axes, it is contained in the closed disk D centred at O and having radius $\frac{d}{2}$. Let Q_{K_*} and Q_D denote the intersection of K_* and D respectively with the quadrant $x \geq 0$, $y \geq 0$. Let the arc of ∂Q_D intercept the x and the y-axes at the points M and N respectively. Let $\text{arc}[M,N]$ denote the closed arc MN of ∂Q_D. Hence $\partial Q_D = [O,M) \cup \text{arc}[M,N] \cup (N,O)$. Since K_* and D have a diameter in common and since K_* is symmetric about the coordinate axes, it follows that Q_{K_*} contains a point on $\text{arc}[M,N]$. We summarize the properties of Q_{K_*} as follows:

(P1) Q_{K_*} is bounded by a line l through $P(\frac{1}{2},1)$ having slope m, $m < 0$.

(P2) Q_{K_*} is bounded by the lines $x = 0$ and $x = \frac{3}{2}$.

(P3) $Q_{K_*} \subseteq Q_D$.

(P4) Q_{K_*} contains a point on $\text{arc}[M,N]$.

Two cases may now be distinguished. Either $P \notin Q_D^\circ$ or $P \in Q_D^\circ$. If $P \notin Q_D^\circ$, we have $0 < \frac{d}{2} \leq \sqrt{(\frac{1}{2})^2 + 1^2} = \frac{1}{2}\sqrt{5}$. In this case we use the area of the disk D to give an upper bound for A. Thus $A \leq \pi(\frac{d}{2})^2$ and so $A/d \leq \frac{\pi}{4}d$. Since $0 < \frac{d}{2} \leq \frac{1}{2}\sqrt{5}$, we have

$$\frac{A}{d} \leq \frac{\pi\sqrt{5}}{4} \approx 1.756 < \alpha. \tag{11.7}$$

If $P \in Q_D^o$, we have $\frac{d}{2} > \frac{1}{2}\sqrt{5}$. Let R_1 be the region $\{(x, y); 0 \leq x \leq \frac{1}{2}, y \geq 1\}$ and let R_2 be the region $\{(x, y); \frac{1}{2} \leq x \leq \frac{3}{2}, 0 \leq y \leq 1\}$. By (P1),(P2) and (P4), l intersects arc$[M, N]$ in R_1 or R_2.

We first observe that if $\frac{d}{2} > \sqrt{(\frac{3}{2})^2 + 1^2} = \frac{1}{2}\sqrt{13}$, then arc$[M, N]$ does not intersect R_2. Hence l must intercept arc$[M, N]$ in R_1 (Figure 11.5). In this case

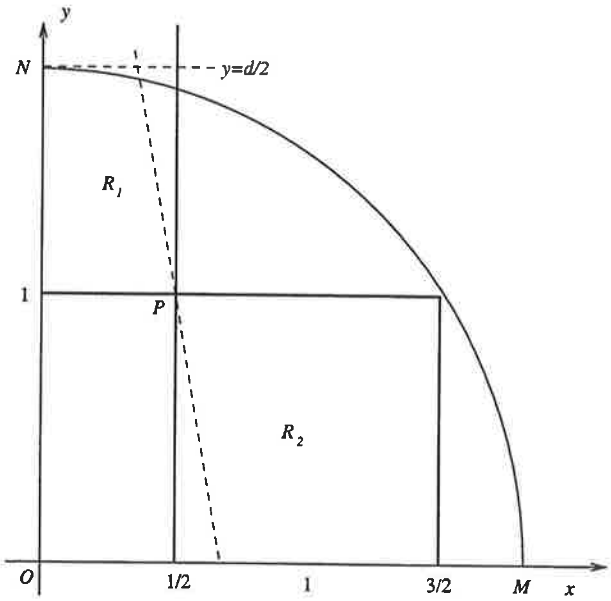


Figure 11.5: The case where $\frac{d}{2} > \frac{1}{2}\sqrt{13}$

$m \leq -(\frac{d}{2}-1)/\frac{1}{2} = -(d-2)$ and Q_{K_*} is contained in the trapezium bounded by the coordinate axes and the lines l and $y = \frac{d}{2}$. Since l has equation $y - 1 = m(x - \frac{1}{2})$, we have

$$\begin{aligned} A \leq 4A(Q_{K_*}) &\leq 4\int_0^{\frac{d}{2}} \left(\frac{1}{m}(y-1) + \frac{1}{2}\right) dy \\ &= \frac{d}{m}(\frac{d}{2} - 2) + d. \end{aligned}$$

Therefore

$$\frac{A}{d} \leq \frac{1}{m}\left(\frac{d}{2} - 2\right) + 1. \tag{11.8}$$

Now if $\frac{d}{2} \geq 2$, then since $m \leq 0$, we have $A/d \leq 1 < \alpha$. If $\frac{1}{2}\sqrt{13} < \frac{d}{2} < 2$, then $\frac{1}{2}\sqrt{13} - 2 < \frac{d}{2} - 2 < 0$. Since $-\infty < m \leq -(d-2) < -1$, we have $-1 < 1/m < 0$. Therefore $A/d \leq 3 - \frac{1}{2}\sqrt{13} \approx 1.2 < \alpha$.

11.5 Proof of Theorem 11.5 for the case $L(1,0)$

From §11.4 we may assume that $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$. For a fixed d, let $a(\theta)$ denote the area of the region S in Q_D containing the origin O and bounded by the lines $x = \frac{3}{2}$ and l making an angle θ, $\theta \in [0, \frac{\pi}{2})$ with the x-axis. Since $Q_{K_*} \subseteq S$, we have $A(Q_{K_*}) \leq A(S)$. We aim to maximize $a(\theta)$.

Let X_1 and X_2 denote the endpoints of the intersection of l with ∂S in the regions R_1 and R_2 respectively. We use the notation $l(\theta)$, $X_1(\theta)$, $X_2(\theta)$ to denote the dependence on θ of l, X_1 and X_2 respectively. Since $a(\theta)$ is continuous in $[0, \frac{\pi}{2}]$, the maximal value of $a(\theta)$ is attained in $[0, \frac{\pi}{2}]$.

Let $s(\theta) = s$ denote the difference $X_1(\theta)P - PX_2(\theta) = X_1P - PX_2$. Let $\delta\theta$ denote a small change in θ and δa the corresponding change in $a(\theta)$. Since $\delta\theta$ is small,

$$\delta a \approx \frac{1}{2}\left((X_1P)^2 - (PX_2)^2\right)\delta\theta.$$

As $\delta\theta \to 0$, $\delta a/\delta\theta \to a'(\theta)$. Hence

$$\begin{aligned} a'(\theta) &= \frac{1}{2}(X_1P - PX_2)(X_1P + PX_2) \\ &= \frac{1}{2}s(\theta)(X_1P + PX_2). \end{aligned}$$

Clearly $a(\theta)$ is differentiable in $(0, \frac{\pi}{2})$. Therefore the maximal value of $a(\theta)$ is attained either at $\theta = 0$, or $\theta = \frac{\pi}{2}$, or at a point $\theta = \theta^*$ where $a'(\theta^*) = 0$. Since

$X_1P + PX_2 > 0$, $a'(\theta) = 0$ only if $s(\theta) = 0$, that is, when P is the midpoint of (X_1, X_2).

We now consider the possible positions of X_1 and X_2 and investigate the conditions for which P is the midpoint of (X_1, X_2). First we note that since X_1 lies in R_1, the x-coordinate of X_1 lies in the interval $[0, \frac{1}{2})$. If P is the midpoint of (X_1, X_2) then X_2 must have x-coordinate in the interval $(\frac{1}{2}, 1]$. Recalling also that at least one of X_1 and X_2 must lie on arc$[M, N]$, we have the following possible positions for X_1 and X_2:

(i) $X_1 \in$ arc$[M, N]$, $X_2 \in [O, M)$. Here the y-coordinate of X_2 is 0. For $P(\frac{1}{2}, 1)$ to be the midpoint of (X_1, X_2), the y-coordinate of X_1 must be 2. This occurs only when $\frac{d}{2} \geq 2$ which lies outside the given range for $\frac{d}{2}$. Hence we may disregard this case.

(ii) $X_1 \in$ arc$[M, N]$, $X_2 \in$ arc$[M, N]$. Here (X_1, X_2) is a chord of D. For $P(\frac{1}{2}, 1)$ to be the midpoint of (X_1, X_2), OP must be perpendicular to (X_1, X_2). Hence the equation of the line X_1X_2 is $y = -\frac{1}{2}x + \frac{5}{4}$. This case arises only when $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{5}{4}$.

(iii) $X_1 \in (N, O)$, $X_2 \in$ arc$[M, N]$. Here the x-coordinate of X_1 is 0. For $P(\frac{1}{2}, 1)$ to be the midpoint of (X_1, X_2), the x-coordinate of X_2 must be 1. This case arises when and only when $\frac{5}{4} < \frac{d}{2} \leq \sqrt{2}$.

It follows from the cases above that if $\sqrt{2} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$, then P is not the midpoint of (X_1, X_2). The ranges of $\frac{d}{2}$ as given by the cases above are shown in Figure 11.6.

We now use these results to help us determine the set for which $a(\theta)$ is maximal for $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$. We prove

Lemma 11.6 *If $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{5}{4}$, then $a(\theta)$ is maximal when l is perpendicular to the line OP. If $\frac{5}{4} < \frac{d}{2} \leq \sqrt{2}$, then $a(\theta)$ is maximal when l intersects ∂Q_D on the lines $x = 0$ and $x = 1$. If $\sqrt{2} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$, then $a(\theta)$ is maximal when l is*

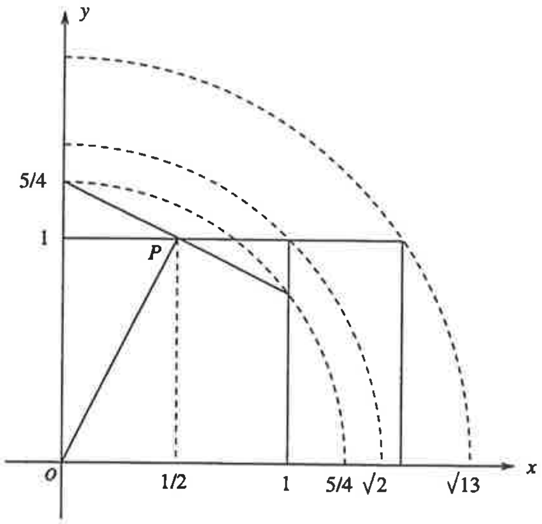


Figure 11.6: The ranges of $\frac{d}{2}$

parallel to the x-axis.

Proof. We first consider the case where $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{5}{4}$ (Figure 11.7). In this case $s(0) > 0$ and $s(\frac{\pi}{2}) < 0$, that is, $a'(0) > 0$ and $a'(\frac{\pi}{2}) < 0$. Hence $a(\theta)$ is maximal at a point $\theta^* \in (0, \frac{\pi}{2})$, that is $s(\theta^*) = 0$. This is the value of θ for which P is the midpoint of (X_1, X_2), that is when l is perpendicular to OP.

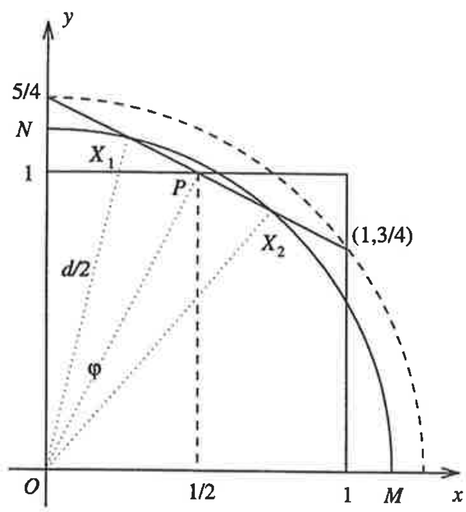


Figure 11.7: The case where $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{5}{4}$

Next we consider the case where $\frac{5}{4} < \frac{d}{2} \leq \sqrt{2}$ (Figure 11.8). We observe that

$s(0) \geq 0$ and $s(\frac{\pi}{2}) < 0$, that is, $a'(0) \geq 0$ and $a'(\frac{\pi}{2}) < 0$. Hence $a(\theta)$ is maximal at a point $\theta^* \in [0, \frac{\pi}{2})$, that is $s(\theta^*) = 0$, or equivalently, when P is the midpoint of (X_1, X_2). In this case l intersects ∂Q_D on the lines $x = 0$ and $x = 1$.

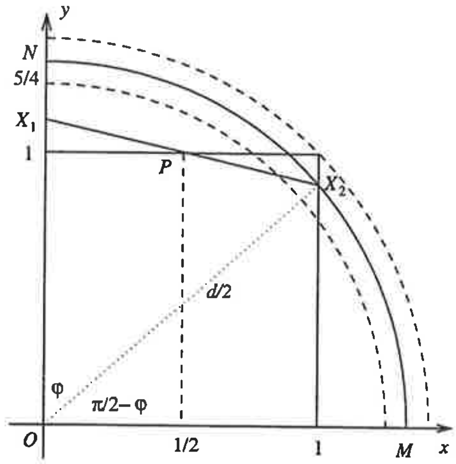


Figure 11.8: The case where $\frac{5}{4} < \frac{d}{2} \leq \sqrt{2}$

Finally suppose that $\sqrt{2} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$ (Figure 11.9). In this case P is not the midpoint of (X_1, X_2) for all positions of X_1 and X_2, that is $s(\theta) \neq 0$ and hence $a'(\theta) \neq 0$ for $\theta \in (0, \frac{\pi}{2})$. It follows that $a(\theta)$ is maximal at $\theta = 0$ or $\theta = \frac{\pi}{2}$. But

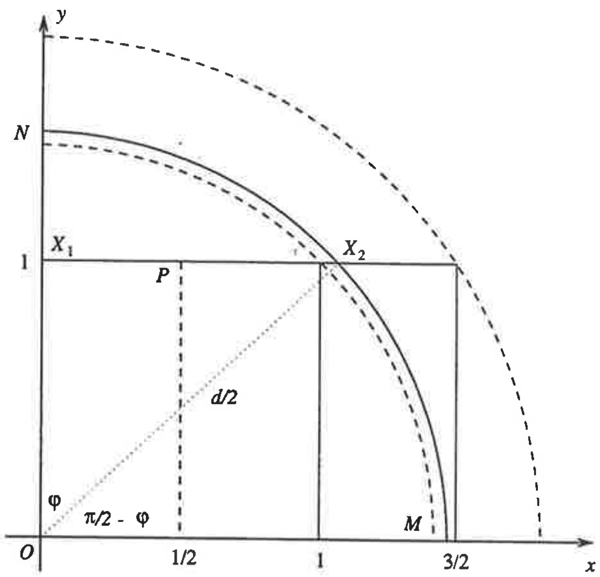


Figure 11.9: The case where $\sqrt{2} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$

$s(0) < 0$ and $s(\frac{\pi}{2}) < 0$; hence $a'(0) < 0$ and $a'(\frac{\pi}{2}) < 0$. Since $a'(\theta) \neq 0$ for $\theta \in (0, \frac{\pi}{2})$, it follows that $a(0) > a(\frac{\pi}{2})$ and the maximal value of $a(\theta)$ is attained when $\theta = 0$, that is, when l is parallel to the x-axis. □

We now use Lemma 11.6 to determine the maximal value of A/d for each of the given ranges of $\frac{d}{2}$. By Lemma 11.6, if $\frac{1}{2}\sqrt{5} < \frac{d}{2} \leq \frac{5}{4}$, the maximal value of $a(\theta)$ occurs when l is perpendicular to the line OP (Figure 11.7). Let $\angle X_1OX_2 = \varphi$ where $0 < \varphi \leq \arcsin\frac{4}{5}$. Calculating the areas of $\triangle X_1OX_2$ and the two sectors making up S, we have

$$\begin{aligned} A = 4A(Q_{K_*}) &\leq 4\left(\frac{1}{2}(\frac{d}{2})^2\sin\varphi + \frac{1}{2}(\frac{d}{2})^2(\frac{\pi}{2} - \varphi)\right) \\ &= \frac{d^2}{2}(\sin\varphi + \frac{\pi}{2} - \varphi). \end{aligned}$$

Noting that $\frac{d}{2} = (\frac{1}{2}\sqrt{5})\sec\frac{\varphi}{2}$, we have

$$\frac{A}{d} \leq \frac{\sqrt{5}}{2}\sec\frac{\varphi}{2}(\sin\varphi + \frac{\pi}{2} - \varphi) = f(\varphi).$$

Differentiating $f(\varphi)$, we get

$$f'(\varphi) = \frac{1}{2}\sqrt{5}\tan\frac{\varphi}{2}\left(-\sin\frac{\varphi}{2} + \frac{1}{2}\sec\frac{\varphi}{2}(\frac{\pi}{2} - \varphi)\right).$$

Solving $f'(\varphi) = 0$ we have $\varphi = 0$ or $\sin\varphi = \frac{\pi}{2} - \varphi$, that is $\varphi = \varphi^* \approx 0.832$. A quick calculation shows that the maximal value of $f(\varphi)$ is attained when $\varphi = \varphi^*$. Hence we have

$$\frac{A}{d} \leq f(\varphi^*) = 2\sqrt{5}\sin\frac{\varphi^*}{2} \approx 1.807 < \alpha. \tag{11.9}$$

By Lemma 11.6, if $\frac{5}{4} < \frac{d}{2} \leq \sqrt{2}$, then the maximal value of $a(\theta)$ occurs when l intersects ∂Q_D on the lines $x = 0$ and $x = 1$ (Figure 11.8). Let $\angle X_1OX_2 = \varphi$.

Therefore, calculating the areas of $\triangle X_1OX_2$ and the sector of angle $\frac{\pi}{2}-\varphi$,

$$\begin{aligned} A = 4A(Q_{K_*}) &\leq 4\left(\frac{1}{2}(\frac{d}{2})^2(\frac{\pi}{2}-\varphi)+1-\frac{1}{2}\cdot\frac{d}{2}\cos\varphi\right) \\ &= d\left(\frac{d}{2}(\frac{\pi}{2}-\varphi)+\frac{4}{d}-\cos\varphi\right) \end{aligned}$$

Noting that $\frac{d}{2}=\operatorname{cosec}\varphi$, we have

$$\frac{A}{d} \leq \operatorname{cosec}\varphi(\frac{\pi}{2}-\varphi)+2\sin\varphi-\cos\varphi = g(\varphi).$$

Since $\frac{5}{4}<\frac{d}{2}\leq\sqrt{2}$, we have $\frac{\pi}{4}\leq\varphi\leq\arcsin\frac{4}{5}$. Differentiating $g(\varphi)$ we obtain

$$g'(\varphi)=\frac{\cos\varphi}{\sin^2\varphi}\left(-\frac{\pi}{2}+\varphi+1-\cos 2\varphi-\frac{1}{2}\sin 2\varphi\right)=0.$$

By solving $g'(\varphi)=0$ for the given range of φ we obtain $\varphi=\varphi^*$ where φ^* is the solution of the equation

$$\frac{1}{2}\sin 2\varphi+\cos 2\varphi=-\frac{\pi}{2}+\varphi+1.$$

This gives $\varphi^*\approx 0.878$. A quick calculation of $g(\frac{\pi}{4})$, $g(\varphi^*)$ and $g(\arcsin\frac{4}{5})$ shows that the maximal value of $g(\varphi)$ is attained when $\varphi=\frac{\pi}{4}$. Hence

$$\frac{A}{d}\leq g(\frac{\pi}{4})\approx 1.818<\alpha. \tag{11.10}$$

By Lemma 11.6, if $\sqrt{2}<\frac{d}{2}\leq\frac{1}{2}\sqrt{13}$, then the maximal value of $a(\theta)$ occurs when l is parallel to the x-axis. Since Q_{K_*} is bounded by $x=\frac{3}{2}$, the following two cases may be distinguished:

(i) $\sqrt{2}<\frac{d}{2}\leq\frac{3}{2}$,

(ii) $\frac{3}{2}<\frac{d}{2}\leq\frac{1}{2}\sqrt{13}$.

We first consider (i) (Figure 11.9). Let $\angle X_1OX_2=\varphi$. Then

$$\begin{aligned} A = 4A(Q_{K_*}) &\leq 4\left(\frac{1}{2}(\frac{d}{2})^2(\frac{\pi}{2}-\varphi)+\frac{1}{2}\cdot\frac{d}{2}\sin\varphi\right) \\ &= d\left(\frac{d}{2}(\frac{\pi}{2}-\varphi)+\sin\varphi\right) \end{aligned}$$

Noting that $\frac{d}{2} = \sec\varphi$, we have

$$\frac{A}{d} \leq \sec\varphi(\frac{\pi}{2} - \varphi) + \sin\varphi = h(\varphi).$$

Since $\sqrt{2} < \frac{d}{2} \leq \frac{3}{2}$, we have $\frac{\pi}{4} < \varphi \leq \arccos\frac{2}{3}$. By differentiating $h(\varphi)$, we get

$$h'(\varphi) = \tan\varphi\left(\sec\varphi(\frac{\pi}{2} - \varphi) - \sin\varphi\right).$$

Solving $h'(\varphi) = 0$ gives $\varphi = 0$ or $\sin 2\varphi = \pi - 2\varphi$ which gives $\varphi = \frac{\pi}{2}$. Both critical values are outside the given range of φ. Hence $h'(\varphi) \neq 0$ for the given range for φ. It may be easily checked that $h(\frac{\pi}{4}) < h(\arccos\frac{2}{3})$. Hence

$$\frac{A}{d} \leq \frac{3}{2}\left(\frac{\pi}{2} - \arccos\frac{2}{3}\right) + \frac{\sqrt{5}}{3} \approx 1.8399 < \alpha. \tag{11.11}$$

Next we consider (ii) (Figure 11.10). Let $x = \frac{3}{2}$ intersect arc$[M,N]$ in the point W. Let $\angle X_1OX_2 = \varphi$ and let OW make an angle of β with the x-axis. Then

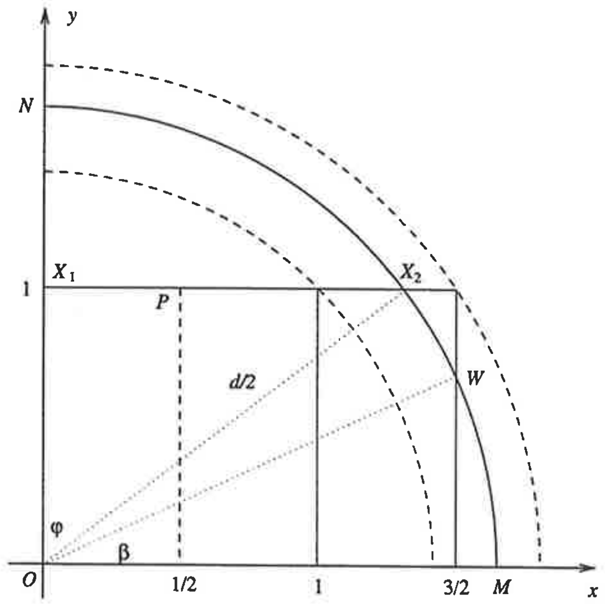


Figure 11.10: The case where $\frac{3}{2} < \frac{d}{2} \leq \frac{1}{2}\sqrt{13}$

$$\begin{aligned} A = 4A(Q_{K_*}) &\leq 4\left(\frac{1}{2}(\frac{d}{2})^2(\frac{\pi}{2} - \varphi - \beta) + \frac{1}{2}\cdot\frac{d}{2}\sin\varphi + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{d}{2}\sin\beta\right) \\ &= d\left(\frac{d}{2}(\frac{\pi}{2} - \varphi - \beta) + \sin\varphi + \frac{3}{2}\sin\beta\right). \end{aligned}$$

Noting that $\frac{d}{2} = \sec\varphi$, we have

$$\frac{A}{d} \leq \sec\varphi(\frac{\pi}{2} - \varphi - \beta) + \sin\varphi + \frac{3}{2}\sin\beta.$$

Furthermore, since $\frac{d}{2} = \frac{3}{2}\sec\beta$, we have $3\cos\varphi = 2\cos\beta$ which gives $\beta = \arccos(\frac{3}{2}\cos\varphi)$. We also note that $\arccos\frac{2}{3} < \varphi \leq \arccos(2/\sqrt{13})$. Substituting $\beta = \arccos(\frac{3}{2}\cos\varphi)$, we obtain the upper bound for A/d,

$$k(\varphi) = \sec\varphi\left(\frac{\pi}{2} - \varphi - \arccos(\frac{3}{2}\cos\varphi)\right) + \sin\varphi + \frac{3}{2}\sqrt{1 - \frac{9}{4}\cos^2\varphi}.$$

The graph of $k(\varphi)$ for the given range of φ is shown in Figure 11.11. A numerical calculation shows that a critical point for k in the given range of φ occurs at $\varphi = \varphi^* \approx 0.847$ (Figure 11.12). Hence

$$A \leq k(\varphi^*) = \alpha \approx 1.841, \tag{11.12}$$

with equality when and only when $K \cong \mathcal{H}_2$.

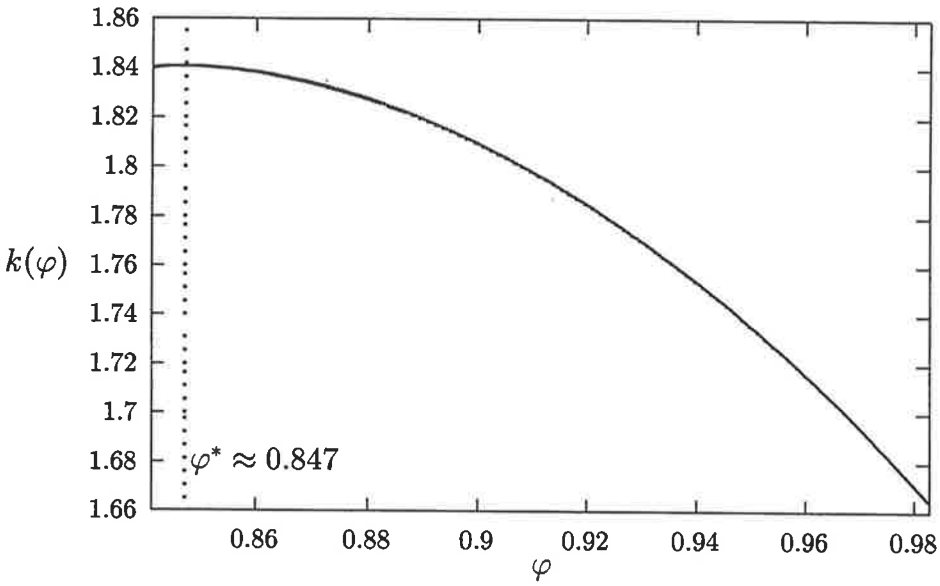


Figure 11.11: The graph of $k(\varphi)$

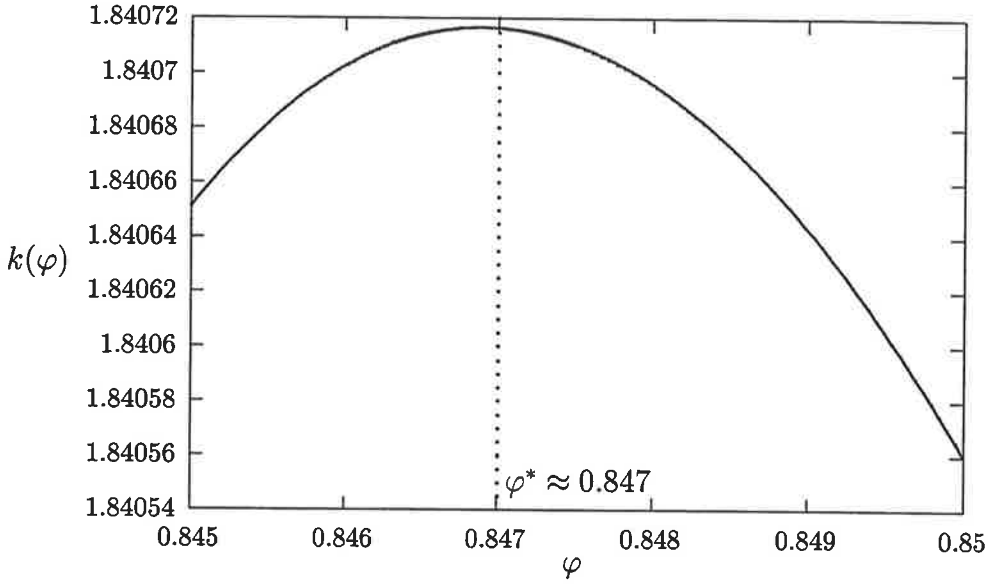


Figure 11.12: The maximal value of $k(\varphi)$ for the case $L(1,0)$

11.6 The cases $L(1,1)$ and $L(z_1,z_2), z_1 \geq 2$

We now complete the proof of Theorem 11.5 by considering the cases $L(1,1)$ and $L(z_1,z_2), z_1 \geq 2$. Here we may remove the symmetry condition on K to establish the following result:

Lemma 11.7 *Let K be a set in $\mathcal{K}^2$ with $G(K^o,\Gamma) = 2$. If A/d is maximal, then the lattice points in K^o are unit distance apart.*

Proof. We will employ Steiner symmetrization as in the proof of Theorem 11.1 to transform K into a set K_* which is symmetric about the lines of symmetrization. We will employ the same notation used in §11.4 and §11.5. We investigate the two cases for L separately:

Case 1: $L(1,1)$. We first symmetrize K with respect to the line $y = 1 - x$ to obtain the closed convex set K_1. Since K^o does not contain any lattice points on the lines $y = \pm 1 + x$ it follows that K^o intersects each of the lines $y = \pm 1 + x$ in the empty set or in line segments of lengths at most $\sqrt{2}$ units. It follows that K_1^o does not contain the points $(\frac{1}{2},\frac{3}{2})$, $(-\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2})$. We now symmetrize K_1 with respect to the line $y = x$ to obtain the closed convex set K_*. Since K_1^o does not contain the points $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{3}{2}, \frac{1}{2})$, K_*^o does not contain these points either. It may be similarly argued that K_*^o does not contain the points $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$ (Figure 11.13).

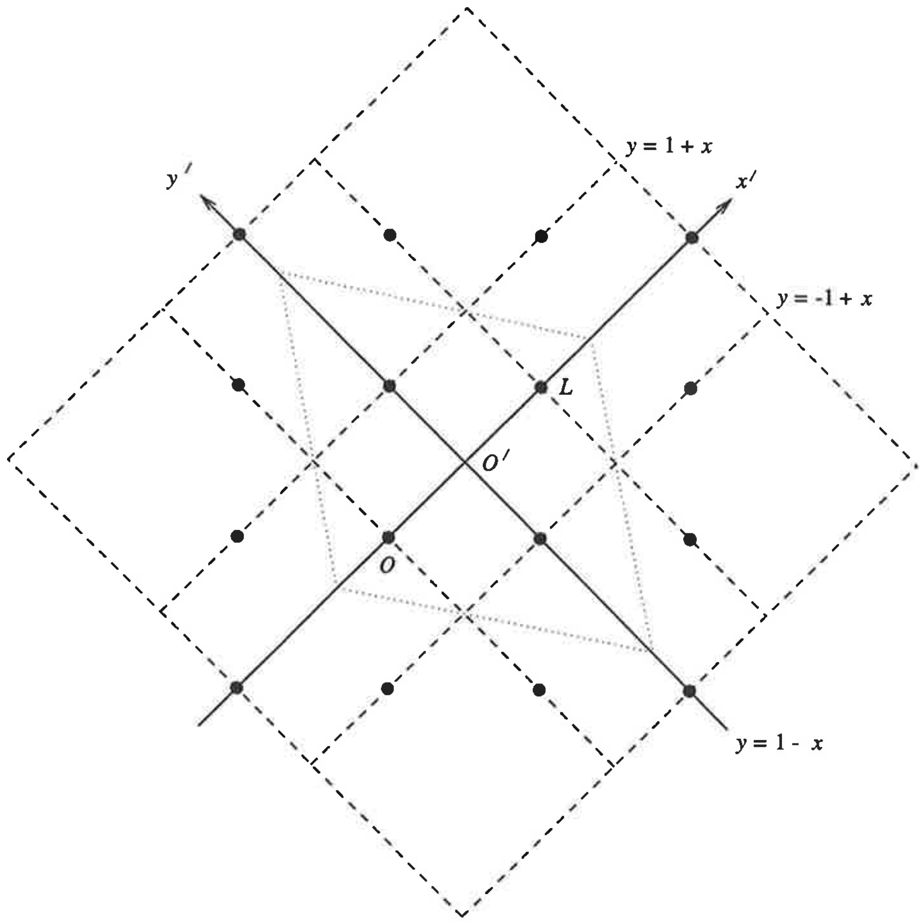


Figure 11.13: The lattice Γ_T'

We now rotate the coordinate axes through an angle of $\frac{\pi}{4}$ in an anticlockwise direction and then translate the origin to the point $M(\frac{1}{2}, \frac{1}{2})$. Clearly K_* is symmetric about the new coordinate axes. We will use $(x, y)'$ to denote a point with coordinates given with respect to the new coordinate axes, x', y'. Since K_*^o does not contain the points $(\frac{1}{2}, \frac{3}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{3}{2}, \frac{1}{2})$, it follows that K_*^o does not contain the points $(\pm\frac{1}{2}\sqrt{2}, \pm\frac{1}{2}\sqrt{2})'$. In particular, K_*^o does not contain the point $P(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})'$. Furthermore, since K_* is convex, it is bounded by a line l containing the point $P(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})'$, and having a non-positive slope m with respect to x' and y'. By considering the lattice Γ'_T having fundamental cell $(\pm\frac{1}{2}\sqrt{2}, \pm\frac{1}{2}\sqrt{2})$, we see that $G(K_*^o, \Gamma'_T) = 0$ (Γ'_T is a translation of Γ'). We may now use (11.1) to obtain $A'/d' \leq \lambda$, where A' and d' are the area and diameter respectively of K_* measured in the scale of Γ'_T. Noting that Γ'_T is essentially an enlargement of Γ with scale factor $\sqrt{2}$, we have $A' = (1/\sqrt{2})^2 A$ and $d' = (1/\sqrt{2})d$. Hence

$$\frac{A}{d} \leq \sqrt{2}\lambda =\approx 1.616 < \alpha.$$

<u>Case 2</u> : $L(z_1, z_2)$ where $z_1 \geq 2$ (see for example Figure 11.14). We first symmetrize K with respect to the x-axis to obtain K_1. Since K^o contains only O and L, the set K_1^o intercepts the lines $x = k$, $k \in \mathbf{Z}$ between the points $(k, \pm 1)$ for $k = 0$ and $k = z_1$ and between the points $(k, \pm\frac{1}{2})$, otherwise. Hence K_1^o intercepts the lines $y = \pm\frac{1}{2}$ in line segments of lengths at most 2 and the line $y = \pm 1$ in line segments of lengths at most 1. We now symmetrize K_1 with respect to the y-axis to obtain K_*. Clearly K_*^o does not contain the points $(\pm\frac{1}{2}, \pm 1)$ and $(\pm 1, \pm\frac{1}{2})$. In particular K_*^o does not contain the points $P(\frac{1}{2}, 1)$ and $P'(1, \frac{1}{2})$. Since K_* is convex, it is bounded by lines l and l' containing the points P and P' respectively with non-positive slopes m and m' respectively. We now have the following two

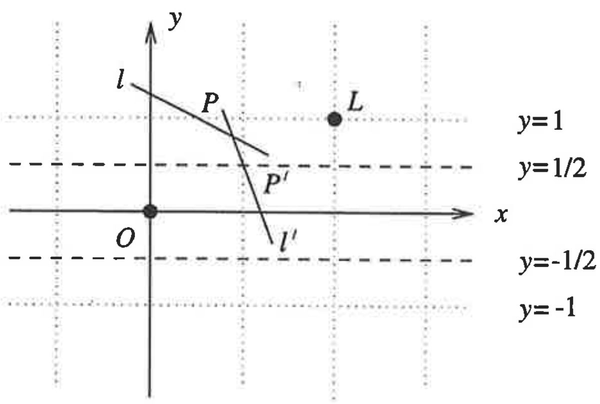


Figure 11.14: The case $L(z_1, z_2)$, $z_1 \geq 2$

cases:

Case a. Suppose that $\frac{d}{2} \leq \frac{3}{2}$. Then since K_* is bounded by l, the inequalities (11.7), (11.9), (11.10) and (11.11) apply. Hence $A/d \leq 1.8399 < \alpha$.

Case b. Suppose that $\frac{d}{2} > \frac{3}{2}$. Then either

(i) Q_{K_*} is bounded by line l having slope $m \leq -1$, or

(ii) Q_{K_*} is bounded by line l' having slope $m' > -1$.

By a reflection about the line $y = x$, it suffices to consider case (i). Here Q_{K_*} is contained in a trapezium bounded by the coordinate axes and the lines l and $y = \frac{d}{2}$. Using (11.8), we have

$$\frac{A}{d} \leq \frac{1}{m}(\frac{d}{2} - 2) + 1.$$

If $\frac{d}{2} \geq 2$, then since $m \leq -1$ we have $A/d \leq 1$. If $\frac{3}{2} < \frac{d}{2} < 2$, have $-\frac{1}{2} < \frac{d}{2} - 2 < 0$. Since also $m \leq -1$, it follows that $A/d < 1.5 < \alpha$. □

By comparing the results of §11.4 and §11.5 and §11.6, we have

$$\frac{A}{d} \leq \alpha \approx 1.841,$$

with equality when and only when $K \cong \mathcal{H}_2$ (Figure 11.3).

Furthermore since $d \leq 2R$, we have

$$\frac{A}{R} \leq 2\alpha \approx 3.682,$$

with equality when and only when $K \cong \mathcal{H}_2$ (Figure 11.3).

Hence Theorem 11.5 is proved.

11.7 Related results and a conjecture

In view of Lemma 11.7, we conjecture a stronger form of Theorem 11.5 by removing the symmetry condition on K.

Conjecture 11.8 *Let K be a non-empty set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 2$. Then*

$$\frac{A}{d} \leq \alpha \approx 1.841,$$

with equality when and only when $K \cong \mathcal{H}_2$ (Figure 11.3).

We end by noting that Theorem 11.1 and Theorem 11.2 have been generalized to arbitrary lattices by Vassallo (1995). Defining λ_1 and $2\mu_1$ as in §6.6, we have

Theorem 11.9 (Vassallo) *Let $K \in \mathcal{K}^2$ and let $G(K^o, \Lambda) = 0$. Then*

$$\frac{A}{d} \leq \alpha = \max\{2\mu_1, 2\sin\frac{\phi}{2}\sqrt{\lambda_1^2 + (2\mu_1)^2}\},$$

where ϕ is the solution of the equation $\sin\theta = \frac{\pi}{2} - \theta$ ($\phi \approx 0.832 \approx 47.7^o$). The bound is best possible.

Corollary 11.10 (Vassallo) *Let K be a non-empty set in $\mathcal{K}^2$ and let $A/d > r\alpha$. Then $G(K^o, \Lambda) \geq r^2$.*

Chapter 12

An area-perimeter inequality for convex sets containing two lattice points

12.1 Introduction

Let K be a set in $\mathcal{K}^2$ with area $A(K) = A$ and perimeter $p(K) = p$. Nosarzewska (1948) proves

Theorem 12.1 (Nosarzewska) *Let K be a set in $\mathcal{K}^2$. Then*

$$A - \frac{1}{2}p < G(K^o, \Gamma). \tag{12.1}$$

The inequality is asymptotically tight.

Nosarzewska's result has been generalized to the case $n = 3$ by Bokowski and Wills (1974) and Schmidt (1972). The result also holds in n dimensions as proved by Bokowski, Hadwiger and Wills (1972).

Bender (1962) proves a special case of Nosarzewska's inequality by letting $G(K^o, \Gamma) = 0$. He proves

Theorem 12.2 (Bender) *Let $K \in \mathcal{K}^2$ and let $G(K^o, \Gamma) = 0$. Then* überflüssig

$$\frac{A}{p} < \frac{1}{2}. \tag{12.2}$$

The limiting infinite strip $\mathcal{P}_0$ (Figure 10.3) *shows that the bound is best possible.*

Since Bender, many new inequalities have been obtained for the functional V/S (Bokowski and Odlyzko 1973; Hadwiger 1970; Hammer 1964, 1966, 1971; Wills 1968, 1970, 1971). Bokowski and Odlyzko (1973) generalize Theorem 12.2 to n dimensions as follows:

Theorem 12.3 (Bokowksi and Odlyzko) *Let $K \in \mathcal{K}^n$ and let ω_n denote the volume of the n-dimensional unit sphere. Suppose that $G(K^o, \Gamma) < g$. Then*

$$\frac{1}{n}(\frac{g}{\omega_n})^{1/n} < \frac{V}{S} < \frac{1}{n}(\frac{g-1}{\omega_n})^{1/n} + \frac{1}{2}, \tag{12.3}$$

where $\omega_n = (\sqrt{\pi})^n / \Gamma(1 + \frac{n}{2})$.

Clearly, when $n = 2$ and $g = 1$, the right hand inequality gives Bender's result. We note also that the above inequality improves results by Hammer (1964, 1966, 1971). Wills (1968, 1970) established the upper bound of (12.3) for the special cases $n = 3$ and $n = 4$ for sets having $G(K^o, \Gamma_n) = 0$ (equivalently $g = 1$). The result by Hadwiger (1970) follows immediately by letting $g = 1$.

Suppose now that $G(K^o, \Gamma) = 1$. Then by Theorem 12.3 we have

$$\frac{A}{p} < \frac{1}{2}\left(\frac{1}{\pi}\right)^{1/2} + \frac{1}{2} \approx 0.782.$$

Scott (1974c) conjectured a tight result for the maximal value of A/p for the class of Γ-admissible sets in $\mathcal{K}^2$ which are symmetric about the origin. The conjecture is proved by Arkinstall and Scott (1979).

Theorem 12.4 (Arkinstall and Scott) *Let K be a Γ-admissible set in $\mathcal{K}^2$ and suppose that K is symmetric about O. Then*

$$\frac{A}{p} \leq 2(2 + \sqrt{\pi})^{-1} \approx 0.530,$$

with equality when and only when K is congruent to the rounded square $\mathcal{U}_1$ shown in Figure 12.1.

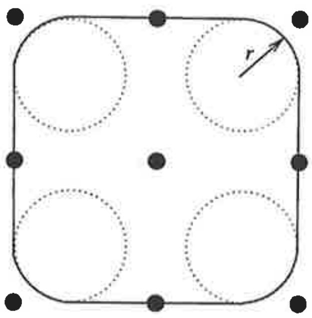


Figure 12.1: The rounded square $\mathcal{U}_1, r \approx 0.530$

This result was obtained independently by Croft (1979) who solved the more general problem of maximising $(A/p)^s$ where s is a parameter between 0 and 2.

In this chapter we prove a result analogous to Theorem 12.4 for a special class of sets in $\mathcal{K}^2$ containing two interior lattice points. We prove

Theorem 12.5 *Let K be a set in $\mathcal{K}^2$. Suppose that $G(K^o, \Gamma) = 2$ and K is symmetric about the midpoint of the line segment joining the two interior lattice points. Then*

$$\frac{A}{p} \leq \lambda \approx 0.636.$$

Equality holds when and only when K is congruent to the rounded hexagon $\mathcal{U}_2$ shown in Figure 12.2.

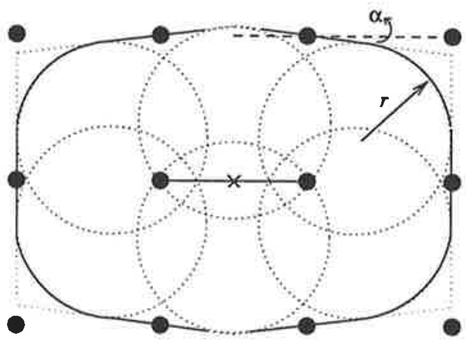


Figure 12.2: The rounded hexagon $\mathcal{U}_2$, $\alpha \approx 0.107 \approx 6.13^o$, $r \approx 0.636$

In §12.2 we describe a general method for finding the maximal value of A/p for sets lying within a given set in $\mathcal{K}^2$. This method is due to Singmaster and

Souppouris (1978). In §12.3, we show how Singmaster's and Souppouris' method helps us solve our lattice point problem. The numerical analysis is carried out in detail in §12.4 and §12.5. In the final section, we conjecture a stronger form of Theorem 12.5 by removing the symmetry condition on K.

12.2 Singmaster's and Souppouris' method

We briefly describe the method employed by Singmaster and Souppouris to determine the maximal value of A/p for sets lying within a bounded, convex, polygonal set, S (this method may be extended to the general convex set by taking the convex set as the limit of inscribed polygons). We will employ the same notation as Singmaster and Souppouris (1978). Hence let r_o denote the inradius of S and let S_r denote the union of all closed disks of radius $r \leq r_o$ contained in S (in other words, S_r is the set S 'rounded off' by circular arcs of radius r which touch the boundary of S). Let S_r' denote the polygon that is formed by the sides of S which S_r touches, produced if necessary. We call S_r' the *contact polygon* of S_r. Let $2\phi_i$ denote the angle of the ith arc of S_r (Figure 12.3).

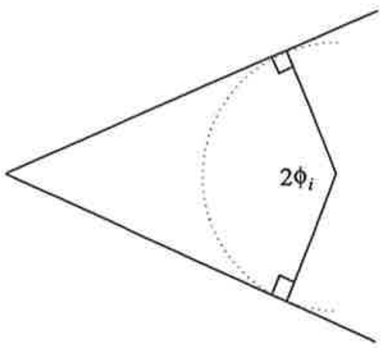


Figure 12.3: The angle of the i-th arc of S_r

Then Singmaster and Souppouris show the following:

(1) $2\phi_i < \pi$, or equivalently, S_r' is a bounded, convex, polygonal set.

(2) The maximal value of A/p for all sets contained in S is attained for a set

S_r where $0 < r \leq r_o$ satisfies the condition

$$\frac{A(S_r)}{p(S_r)} = r. \tag{12.4}$$

(3) The area and the perimeter of S_r are given by the formulae:

$$A(S_r) = A(S_r') - kr^2, \tag{12.5}$$

$$p(S_r) = p(S_r') - 2kr, \tag{12.6}$$

where $k = \sum_i \tan \phi_i - \pi$

Substituting (12.5) and (12.6) into (12.4), we have

$$r = \frac{A(S_r') - r^2 k}{p(S_r') - 2rk}.$$

This simplifies to give the following quadratic equation in r:

$$kr^2 - rp(S_r') + A(S_r') = 0.$$

Since $p(S_r) = p(S_r') - 2rk > 0$, we have $r < p(S_r')/(2k)$. Hence solving the quadratic equation for r, we have

$$r = r^* = \frac{1}{2k}\left(p(S_r') - \sqrt{(p(S_r'))^2 - 4kA(S_r')}\right). \tag{12.7}$$

Since S is a bounded, convex polygon, there is a finite number of possible contact polygons. As described by Singmaster and Souppouris, we begin with $r = \varepsilon$ where ε is a small positive number and we consider the corresponding contact polygon S_r'. The value of r^* may then be evaluated using (12.7). If this value of r^* gives a set S_{r^*} for which the contact polygon is the contact polygon of S_r, then we are done. Otherwise we proceed to the next possible contact polygon. We repeat the process until we arrive at a value for r for which $S_{r^*}' = S_r'$.

12.3 Some preliminaries to the lattice point problem

Let K now be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 2$. Let O and $L(z_1, z_2)$ be the lattice points contained in K^o. Let M denote the midpoint of OL. As in §11.4, we will consider the following cases for L:

(i) $L(1,0)$ (ii) $L(1,1)$ (iii) $L(z_1, z_2)$, $z_1 \geq 2$

We examine each case separately in the next three sections and show that case (i) gives the maximal result. In cases (i) and (ii), we employ Steiner symmetrization on K about a given line l through M, to transform the set K into convex set K_*. We recall from Theorem 2.3 that Steiner symmetrization preserves convexity and areas and does not increase perimeters. Hence $A(K)/p(K) \leq A(K_*)/p(K_*)$. We note that the set K_* may contain more than two interior lattice points. Nevertheless, it will be sufficient for us to prove Theorem 12.5 for the set K_*. For a simpler notation, we let $A(K_*) = A$ and $p(K_*) = p$.

Since K is symmetric about M, K_* is also symmetric about the line l' through M perpendicular to l. By taking l' and l to be the new x and y-axes respectively, it will be shown that K_* satisfies the following properties:

(P1) K_* is convex and symmetric about the new coordinate axes.

(P2) K_* is bounded by the lines $x = \pm x_o$, where x_o is a number to be specified.

(P3) K_* is contained in a rhombus with edges each containing a fixed point (not necessarily a lattice point). We shall see that these fixed points are symmetrically placed about the coordinate axes and arise because of the lattice constraint on K. Let P_1, P_2, P_3 and P_4 denote these fixed points. We will specify the point P_1 with coordinates (x, y) with respect to the new coordinate axes.

Now let p_1, p_2, p_3 and p_4 be the edges of the rhombus containing the fixed points P_1, P_2, P_3 and P_4 respectively, with each edge making an acute angle of α with the x-axis. Let $R(\alpha)$ denote the rhombus determined by the lines p_i, $i = 1, \ldots, 4$ and let $H(\alpha)$ denote the hexagon determined by the lines p_i, $i = 1, \ldots, 4$ and $x = \pm x_o$. If p_1 intersects the line $x = x_o$ in $y \leq 0$, then K_* is contained in $R(\alpha)$ (Figure 12.4a). Otherwise, by (P2), K_* is contained in $H(\alpha)$ (Figure 12.4b).

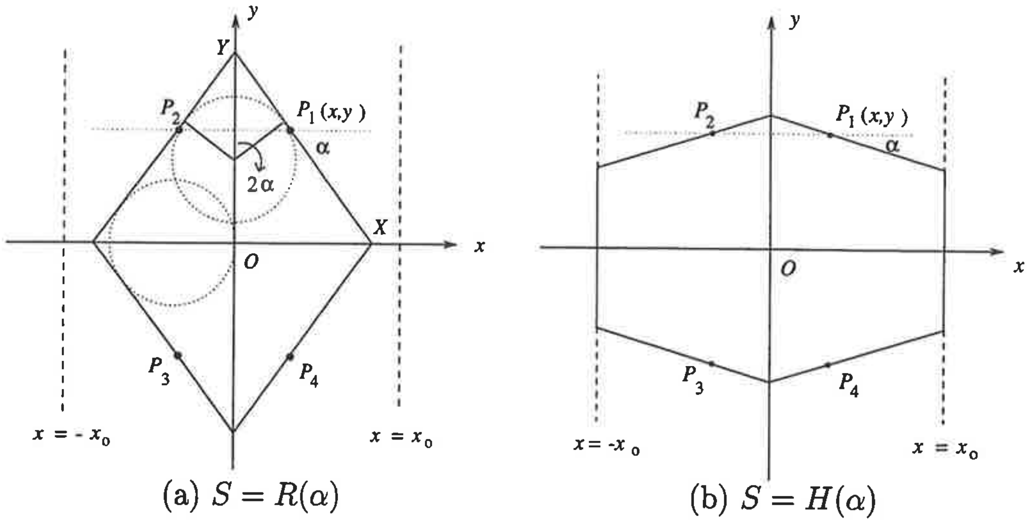


Figure 12.4: Bounding sets for K_* for the lattice point problem

The problem then becomes one of maximizing A/p over all sets contained in S where S is $R(\alpha)$ or $H(\alpha)$. We now use Singmaster's and Souppouris' method to formulate our problem.

If $S = R(\alpha)$, then clearly $S_r' = R(\alpha)$ (Figure 12.4a). If, on the other hand, $S = H(\alpha)$, the following lemma shows that S_r' is either $R(\alpha)$ or $H(\alpha)$.

Lemma 12.6 *Let $S = H(\alpha)$ be the hexagon $A_1A_2A_3A_4A_5A_6$, with $\angle A_1 = \angle A_4$ and $\angle A_2 = \angle A_3 = \angle A_5 = \angle A_6$. Let b_2 and b_3 denote the angle bisectors of $\angle A_2$ and $\angle A_3$ respectively and let T be the point of intersection of b_2 and b_3. Let t denote the distance of T from the line A_2A_3. If $r \leq t$, then $S_r' = S = H(\alpha)$. If $r > t$, then $S_r' = R(\alpha)$.*

Proof. Due to the symmetry of S, the incentre of S is at the point O, having distance x_o from the line A_2A_3. Suppose first that $t \geq x_o$ (Figure 12.5). Clearly, $r_o = x_o \leq t$. Hence any disk having radius $r \leq r_o \leq t$ may be positioned on b_2 to touch the sides A_1A_2 and A_2A_3. By symmetry, it follows that $S_r' = S = H(\alpha)$.

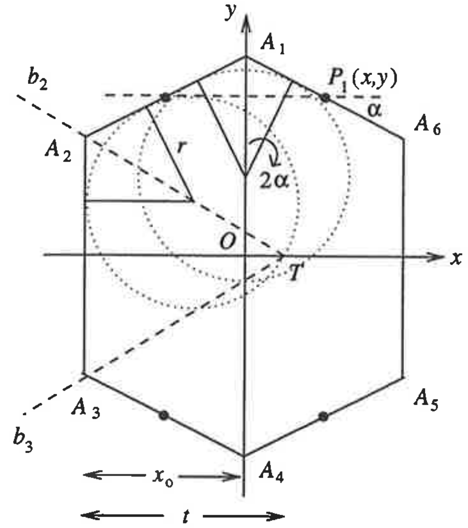


Figure 12.5: The case where $t \geq x_o$

Now suppose that $t < x_o$. Clearly then, $r_o > t$. If $r < t < r_o$, then as before, any disk having radius $r < t < r_o$ may be positioned on b_2 to touch the sides A_1A_2 and A_2A_3. By symmetry, it follows that $S_r' = S = H(\alpha)$ (Figure 12.6).

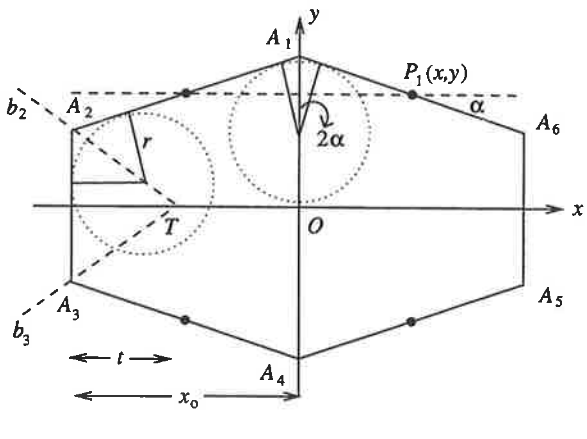


Figure 12.6: The case where $t < x_o$ and $r \leq t$

If, on the other hand, $t < r \leq r_o$, then any disk in S having radius r does not touch the edge A_2A_3. By symmetry, it is clear that $S_r' = R(\alpha)$ (Figure 12.7).

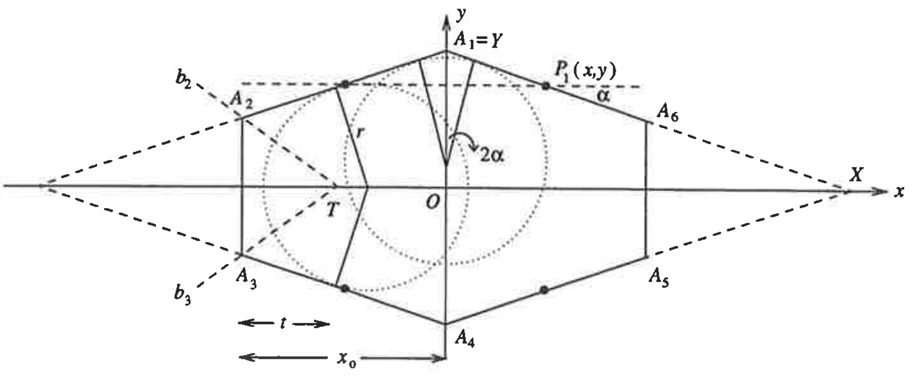


Figure 12.7: The case where $t < x_o$ and $r > t$

Hence, if $r \leq t$, then $S_r' = S = H(\alpha)$. If $r > t$, then $S_r' = R(\alpha)$. □

We now formulate the two cases for S_r' separately.

Case 1. $S_r' = R(\alpha)$ (Figures 12.4a and 12.7). Let p_1 intersect the x and y-axes in the points X and Y respectively. Then $OX = x + y/\tan\alpha$, $OY = y + x\tan\alpha$ and $XY = y/\sin\alpha + x/\cos\alpha$. Since $A(S_r') = 4A(\triangle OXY)$ and $p(S_r') = 4.XY$, we have

$$\left.\begin{array}{rcl} A(S_r') &=& 4xy + 2x^2\tan\alpha + \dfrac{2y^2}{\tan\alpha}, \\ p(S_r') &=& 4\left(\dfrac{y}{\sin\alpha} + \dfrac{x}{\cos\alpha}\right), \\ k &=& 2\tan\alpha + \dfrac{2}{\tan\alpha} - \pi. \end{array}\right\} \qquad (12.8)$$

Using (12.7), we have $r^* = r_R^*(\alpha)$.

Case 2. $S_r' = H(\alpha)$ (Figures 12.5 and 12.6). Here we have $A_1A_2 = x_o/\cos\alpha$ and $A_2A_3 = 2y - 2(x_o - x)\tan\alpha$. By taking $A(S_r')$ to be the sum of the areas of the rectangle $A_2A_3A_5A_6$ and the triangles $A_1A_2A_6$ and $A_3A_4A_5$, and $p(S_r')$ to be $2.A_2A_3 + 4.A_1A_2$, we have

$$\left.\begin{array}{rcl} A(S_r') &=& 4x_o(y-(x_o-x)\tan\alpha)+2x_o^2\tan\alpha, \\ p(S_r') &=& 4(y-(x_o-x)\tan\alpha)+\dfrac{4x_o}{\cos\alpha}, \\ k &=& 2\tan\alpha+4\tan(\dfrac{\pi}{4}-\dfrac{\alpha}{2})-\pi \\ &=& \dfrac{4}{\cos\alpha}-2\tan\alpha-\pi. \end{array}\right\} \tag{12.9}$$

Using (12.7), we obtain $r^* = r_H^*(\alpha)$. By Lemma 12.6, Case 2 applies only when $r_H^*(\alpha) \le t = t(\alpha)$, where

$$t = t(\alpha) = (y-(x_o-x)\tan\alpha)\tan(\frac{\pi}{4}+\frac{\alpha}{2}). \tag{12.10}$$

Otherwise, we have Case 1 and $r^* = r_R^*(\alpha)$.

We now define the function $f(\alpha) = f$ as follows:

$$f(\alpha) = \begin{cases} r_H^*(\alpha), & \text{if } r_H^*(\alpha) \le t(\alpha), \\ r_R^*(\alpha), & \text{otherwise} \end{cases} \tag{12.11}$$

We seek to find the maximal value of $f(\alpha)$ for $0 \le \alpha < \frac{\pi}{2}$. We will use the notation and results in this section in §12.4 and §12.5.

12.4 The case $L(1,0)$

In this case, the centre of K is the point $M(\frac{1}{2},0)$. We first symmetrize K with respect to l where l is the line $x=\frac{1}{2}$ to obtain the set K_*. As already described in §11.4, the set K_* has properties (P1), (P2) and (P3), where $x_o = \frac{3}{2}$ and P_1 is the point with coordinates $(\frac{1}{2},1)$ (Figure 12.8). By substituting $x=\frac{1}{2}$, $y=1$ and $x_o=\frac{3}{2}$ into (12.8) and (12.9), we have

Case 1. $S_r' = R(\alpha)$.

$$\begin{array}{rcl} A(S_r') &=& 2+\dfrac{2}{\tan\alpha}+\dfrac{1}{2}\tan\alpha, \\ p(S_r') &=& \dfrac{4}{\sin\alpha}+\dfrac{2}{\cos\alpha}, \\ k &=& 2\tan\alpha+\dfrac{2}{\tan\alpha}-\pi. \end{array}$$

Using (12.7), we have $r^* = r_R^*(\alpha)$.

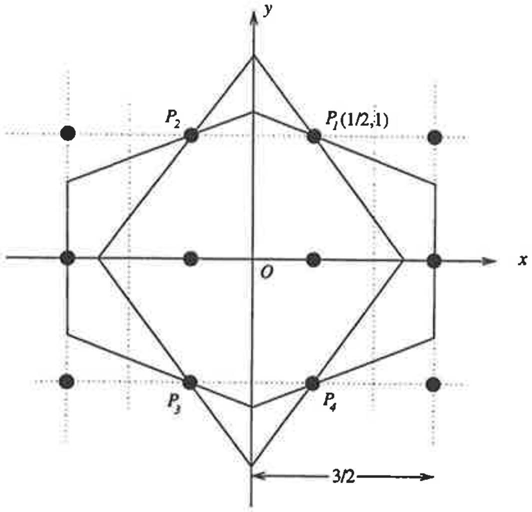


Figure 12.8: Bounding sets for K_* for the case $L(1,0)$

<u>Case 2</u>. $S'_r = H(\alpha)$.

$$\begin{aligned}
A(S'_r) &= 6 - \frac{3}{2}\tan\alpha, \\
p(S'_r) &= \frac{6}{\cos\alpha} - 4\tan\alpha + 4, \\
k &= \frac{4}{\cos\alpha} - 2\tan\alpha - \pi.
\end{aligned}$$

Using (12.7), we obtain $r^* = r^*_H(\alpha)$.

Now from (12.10), we have

$$t = t(\alpha) = (1 - \tan\alpha)\tan(\frac{\pi}{4} + \frac{\alpha}{2}).$$

Solving $r^*_H(\alpha) \leq t(\alpha)$, we have $\alpha \leq \alpha^* \approx 0.601$. Hence for $\alpha \leq \alpha^*$, $S'_r = H(\alpha)$, otherwise $S'_r = R(\alpha)$. From (12.11), we therefore have

$$f(\alpha) = \begin{cases} r^*_H(\alpha), & \alpha \leq \alpha^* \approx 0.601, \\ r^*_R(\alpha), & \alpha > \alpha^* \approx 0.601, \end{cases} \tag{12.12}$$

The graphs of $r^*_R(\alpha)$ and $r^*_H(\alpha)$ are shown in Figure 12.9. The value $\alpha = \alpha^* \approx 0.601$ indicates a transition from the contact polygon $H(\alpha)$ to the contact polygon $R(\alpha)$. The graph of f is therefore obtained by taking the relevant parts of the graphs of $r_R^*(\alpha)$ and $r_H^*(\alpha)$ as defined in (12.12). A numerical calculation shows that the maximal value of f occurs where $\alpha = \alpha^{**} \approx 0.107$ and $f(\alpha^{**}) = \lambda \approx 0.636$ (Figure 12.10). ← acceptable or not ??

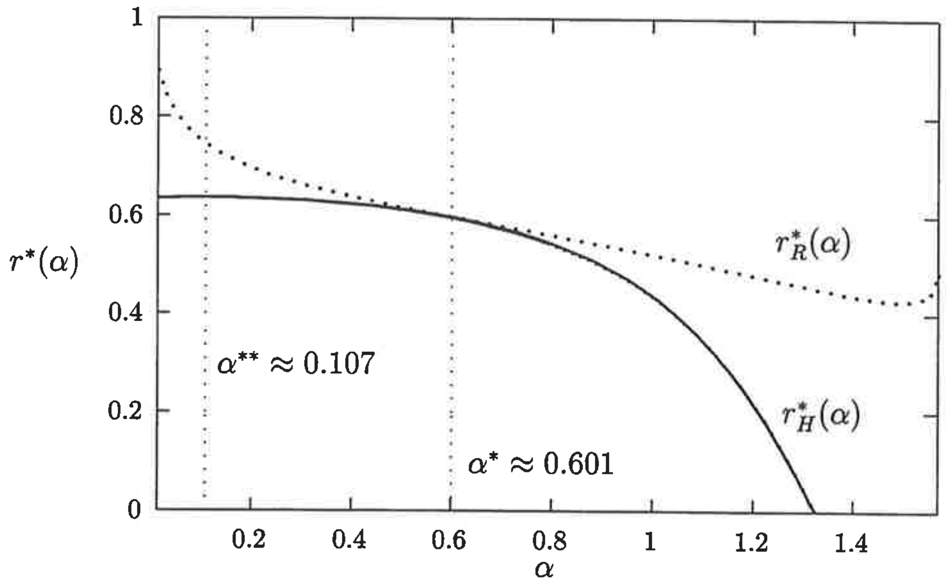


Figure 12.9: The graph of $f(\alpha)$ for the case $L(1, 0)$

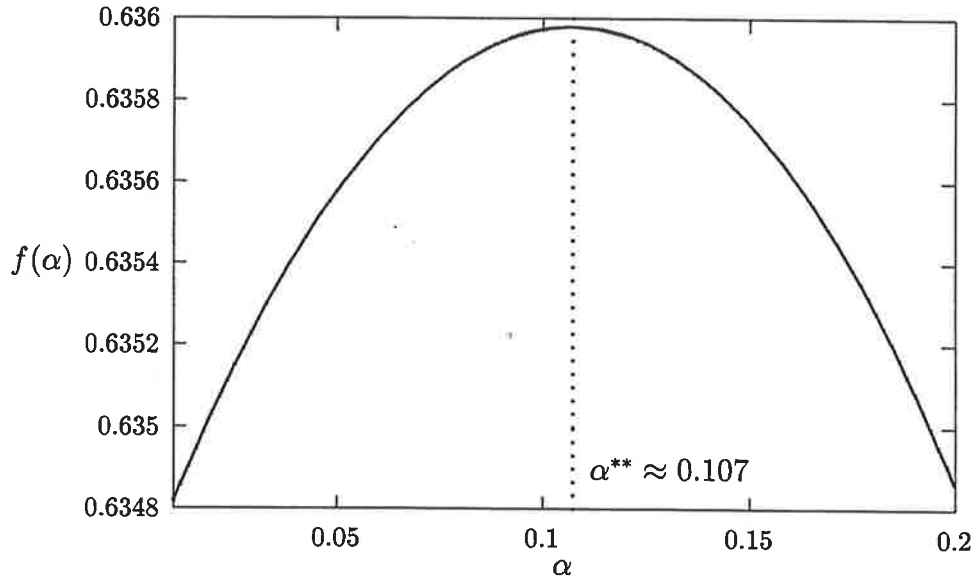


Figure 12.10: The maximal value of $f(\alpha)$ for the case $L(1, 0)$

12.5 The case $L(1,1)$

Since K is symmetric about $M(\frac{1}{2}, \frac{1}{2})$, K is bounded by the parallel lines x_1 and x_2 containing the points $X_1(0,1)$ and $X_2(1,0)$ respectively, having positive slope m. By a reflection about the line $y = x$, it suffices to consider those cases for which $m \geq 1$. We now consider the following two cases for m.

Case a. $m \geq 2$ (Figure 12.11). Here we symmetrize K with respect to l where l is the line $x = \frac{1}{2}$. Since $m \geq 2$, the distance between x_1 and x_2 in a direction parallel to the x-axis is at most $\frac{3}{2}$. Hence the symmetrized set K_* lies in the parallel strip bounded by the lines $x = -\frac{1}{4}$ and $x = \frac{5}{4}$. Furthermore, since K^o contains no point of Γ on the lines $y = 2$ and $y = -1$, K^o intersects these lines in line segments of lengths at most one. Clearly then K^o_* does not contain the points with coordinates $(1,2)$, $(0,2)$, $(0,-1)$ and $(1,-1)$.

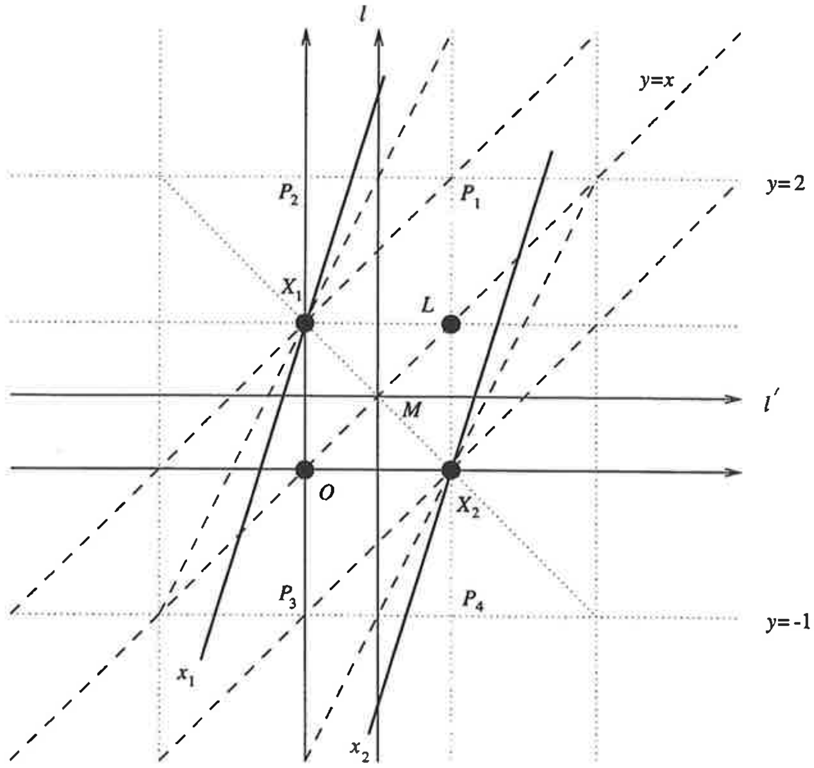


Figure 12.11: The case where $m \geq 2$

It may be easily checked that K_* satisfies properties (P1), (P2) and (P3), where $x_o = \frac{3}{4}$ and P_1 is the point with coordinates $(\frac{1}{2}, \frac{3}{2})$ (Figure 12.12).

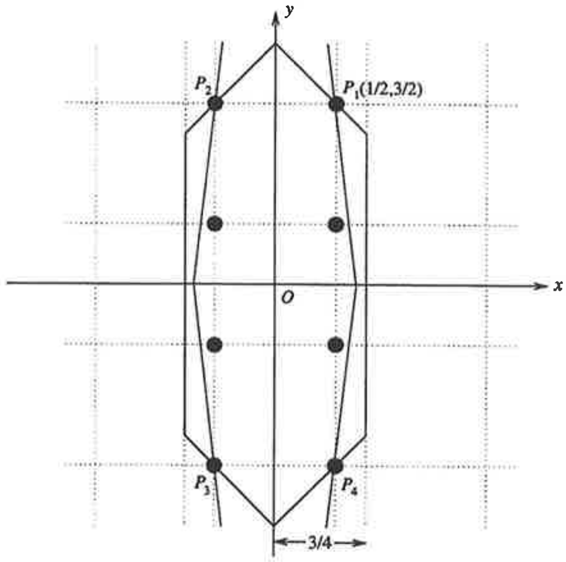


Figure 12.12: Bounding sets for K_* for the case $m \geq 2$

By substituting $x = \frac{1}{2}$, $y = \frac{3}{2}$ and $x_o = \frac{3}{4}$ into (12.8) and (12.9), we have

Case 1. $S'_r = R(\alpha)$.

$$\begin{aligned} A(S'_r) &= 3 + \frac{1}{2}\tan\alpha + \frac{9}{2\tan\alpha}, \\ p(S'_r) &= \frac{6}{\sin\alpha} + \frac{2}{\cos\alpha}, \\ k &= 2\tan\alpha + \frac{2}{\tan\alpha} - \pi. \end{aligned}$$

Using (12.7), $r = r^*_R(\alpha)$ may be found.

Case 2. $S'_r = H(\alpha)$.

$$\begin{aligned} A(S'_r) &= \frac{9}{2} + \frac{3}{8}\tan\alpha, \\ p(S'_r) &= 6 + \frac{3}{\cos\alpha} - \tan\alpha, \\ k &= \frac{4}{\cos\alpha} - 2\tan\alpha - \pi. \end{aligned}$$

Using (12.7), $r^* = r^*_H(\alpha)$ may be found.

Now from (12.10), we have

$$t = t(\alpha) = \Big(\frac{3}{2} - \frac{1}{4}\tan\alpha\Big)\tan(\frac{\pi}{4} + \frac{\alpha}{2}).$$

Solving $r_H^*(\alpha) \le t(\alpha)$, we have $\alpha \le \alpha^* \approx 1.401$. Hence for $\alpha \le \alpha^*$, $S_r' = H(\alpha)$, otherwise $S_r' = R(\alpha)$. From (12.11), we therefore have

$$f(\alpha) = \begin{cases} r_H^*(\alpha), & \alpha \le \alpha^* \approx 1.401, \\ r_R^*(\alpha), & \alpha > \alpha^* \approx 1.401, \end{cases} \tag{12.13}$$

The graphs of $r_R^*(\alpha)$ and $r_H^*(\alpha)$ are shown in Figure 12.13. The value $\alpha = \alpha^* \approx 1.401$ indicates a transition from the contact polygon $H(\alpha)$ to the contact polygon $R(\alpha)$. The graph of f is therefore obtained by taking the relevant parts of the graphs of $r_R^*(\alpha)$ and $r_H^*(\alpha)$ as defined in (12.13). From the graph it is seen that $f(\alpha) < \lambda \approx 0.636$.

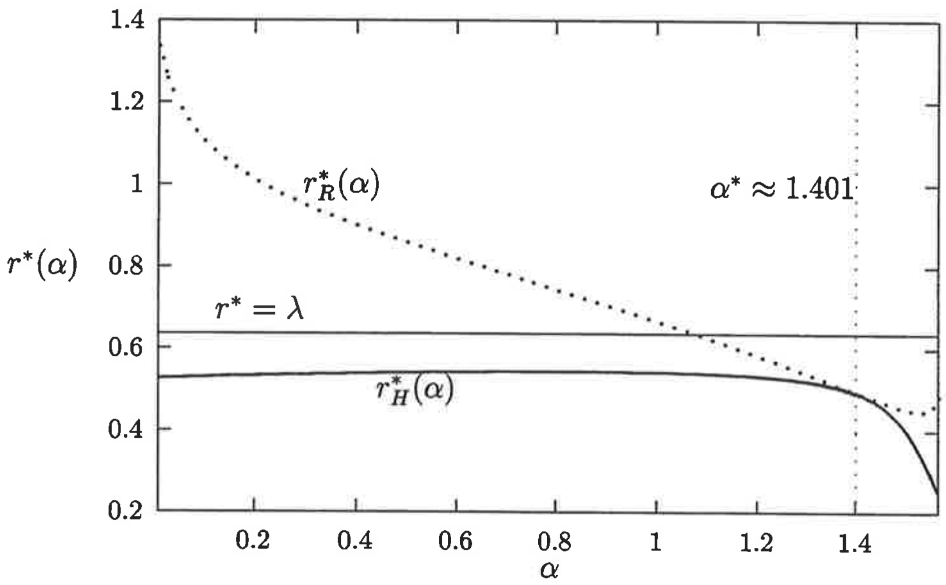


Figure 12.13: The graph of $f(\alpha)$ for the case $L(1,1)$, $m \ge 2$

Case b. $1 \le m < 2$ (Figure 12.14). We first show that if K is a maximal set, it lies in the parallel strip bounded by the lines $y = \pm 3 + x$. Suppose that K extends beyond the line $y = -3 + x$. Then there is a point $X \in K$ which lies in $y < -3 + x$. Since $m < 2$, X also lies in $y > -2 + 2x$. Now clearly

$XM > \sqrt{(9/2)^2 + (3/2)^2} > 9/2$. Hence $p(K) > 18$. By (12.1), we have

$$\frac{A}{p} < \frac{2}{p} + \frac{1}{2} < \frac{2}{18} + \frac{1}{2} \approx 0.611 < \lambda.$$

Hence K is not maximal. We may therefore assume that K is bounded by the lines $y = \pm 3 + x$.

We now symmetrize K with respect to the line l where l is the line $y = 1 - x$. Since K is symmetric about $M(\frac{1}{2}, \frac{1}{2})$, K is bounded by parallel lines containing the points with coordinates $(-1,-1)$ and $(2,2)$. Hence K_* is bounded by the lines $y = 4 - x$ and $y = -2 - x$. Furthermore since K^o does not contain any point of Γ on the lines $y = 1 + x$ and $y = -1 + x$, it intersects these lines in line segments of lengths at most $\sqrt{2}$. It follows that K_*^o intersects these lines in line segments of lengths at most $\sqrt{2}$. Clearly then, K_*^o does not contain the points with coordinates $(\frac{1}{2}, \frac{3}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{2}, \frac{1}{2})$. It may be easily checked that K_* satisfies properties (P1), (P2) and (P3), where $x_o = \frac{3}{2}\sqrt{2}$ and P_1 is the point with coordinates $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ (Figure 12.15).

Since K_* is also bounded by the lines $y = \pm\frac{3}{2}\sqrt{2}$, we need only consider the case where $0 \le \alpha \le \frac{\pi}{4}$. By substituting $x = \frac{1}{2}\sqrt{2}$, $y = \frac{1}{2}\sqrt{2}$ and $x_o = \frac{3}{2}\sqrt{2}$ into (12.8) and (12.9), we have

Case 1. $S_r' = R(\alpha)$.

$$\begin{aligned} A(S_r') &= 2 + \frac{1}{\tan\alpha} + \tan\alpha, \\ p(S_r') &= 2\sqrt{2}(\frac{1}{\cos\alpha} + \frac{1}{\sin\alpha}), \\ k &= 2\tan\alpha + \frac{2}{\tan\alpha} - \pi. \end{aligned}$$

Using (12.7), $r = r_R^*(\alpha)$ may be found.

Case 2. $S_r' = H(\alpha)$.

$$\begin{aligned} A(S_r') &= 6 - 3\tan\alpha, \\ p(S_r') &= \left(\frac{6}{\cos\alpha} - 4\tan\alpha + 2\right)\sqrt{2}, \end{aligned}$$

$$k = \frac{4}{\cos\alpha} - 2\tan\alpha - \pi.$$

Using (12.7), $r = r_H^*(\alpha)$ may be found.

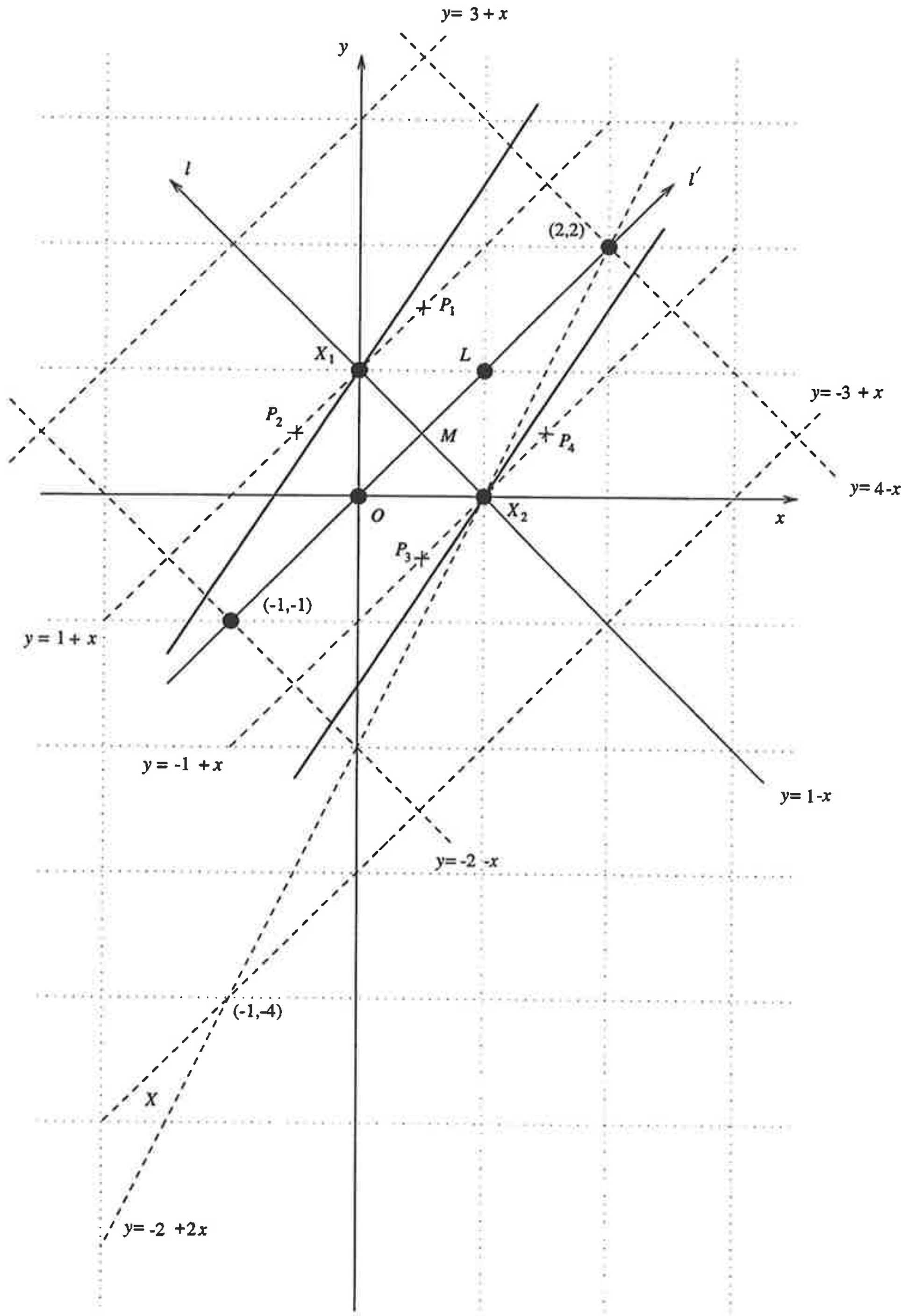


Figure 12.14: The case where $1 \leq m < 2$

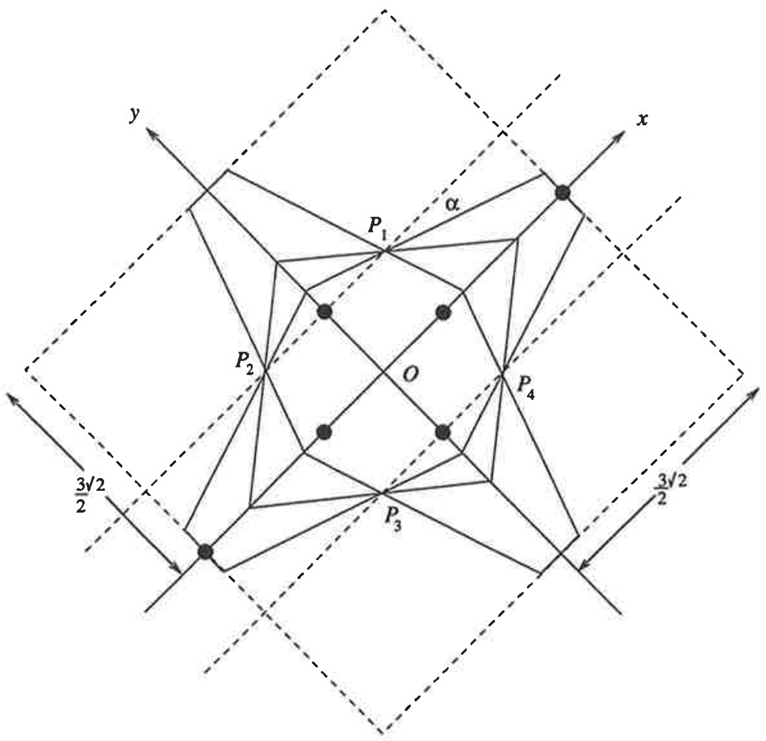


Figure 12.15: Bounding sets for K_* for the case $1 \leq m < 2$

Now from (12.10), we have

$$t = t(\alpha) = \left(\frac{\sqrt{2}}{2} - \sqrt{2}\tan\alpha\right)\tan(\frac{\pi}{4} + \frac{\alpha}{2}).$$

Solving $r^*_H(\alpha) \leq t(\alpha)$, we have $\alpha \leq \alpha^* \approx 0.180$. Hence for $\alpha \leq \alpha^*$, $S'_r = H(\alpha)$, otherwise $S'_r = R(\alpha)$. From (12.11), we therefore have

$$f(\alpha) = \begin{cases} r^*_H(\alpha), & \alpha \leq \alpha^* \approx 0.180, \\ r^*_R(\alpha), & \alpha > \alpha^* \approx 0.180, \end{cases} \tag{12.14}$$

The graphs of $r^*_R(\alpha)$ and $r^*_H(\alpha)$ are shown in Figure 12.16. The value $\alpha = \alpha^* \approx 0.180$ indicates a transition from the contact polygon $H(\alpha)$ to the contact polygon $R(\alpha)$. The graph of f is therefore obtained by taking the relevant parts of the graphs of $r^*_R(\alpha)$ and $r^*_H(\alpha)$ as defined in (12.14). From the graph it is seen that $f(\alpha) < \lambda \approx 0.636$.

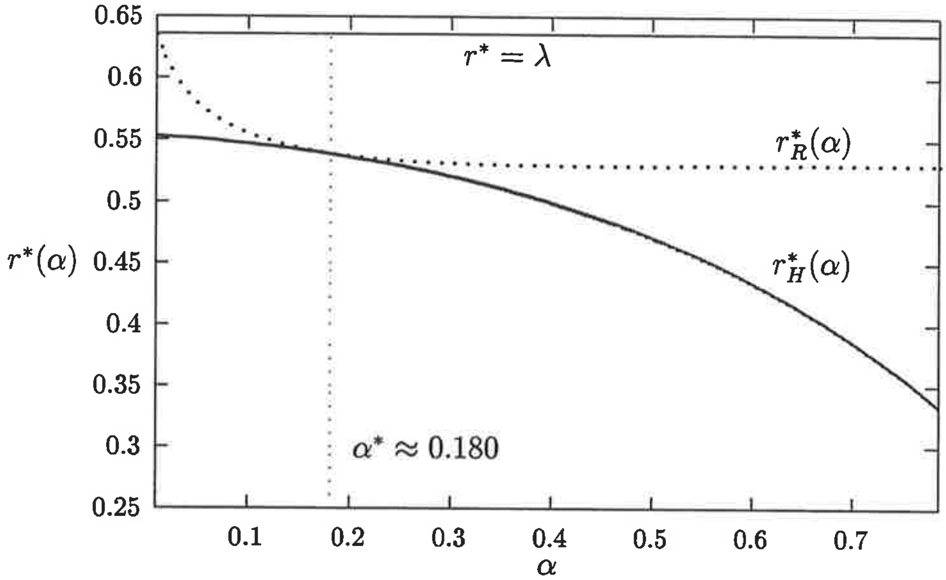


Figure 12.16: The graph of $f(\alpha)$ for the case $L(1, 1)$, $1 \leq m < 2$

12.6 The case $L(z_1, z_2)$, $z_1 \geq 2$

From §2.5, we need only consider those cases for which $z_1 \geq z_2$ and z_1 and z_2 are relatively prime. We consider the following three cases:

(i) z_1 odd and z_2 odd. Let X and Y be the points with coordinates $(\frac{1}{2}(z_1 - 1), \frac{1}{2}(z_2 - 1))$ and $(\frac{1}{2}(z_1 + 1), \frac{1}{2}(z_2 + 1))$ respectively.

(ii) z_1 odd and z_2 even. Let X and Y be the points with coordinates $(\frac{1}{2}(z_1 - 1), z_2)$ and $(\frac{1}{2}(z_1 + 1), z_2)$ respectively.

(iii) z_1 even and z_2 odd. Let X and Y be the points with coordinates $(z_1, \frac{1}{2}(z_2 - 1))$ and $(z_1, \frac{1}{2}(z_2 + 1))$ respectively.

In all three cases, since K is symmetric about $M(\frac{1}{2}z_1, \frac{1}{2}z_2)$, K is contained in a parallel strip bounded by the lines x and y having the same positive slope and having width at most 1 (Figure 12.17). By Theorem 12.2, we have

$$\frac{A}{p} < \frac{1}{2} < \lambda.$$

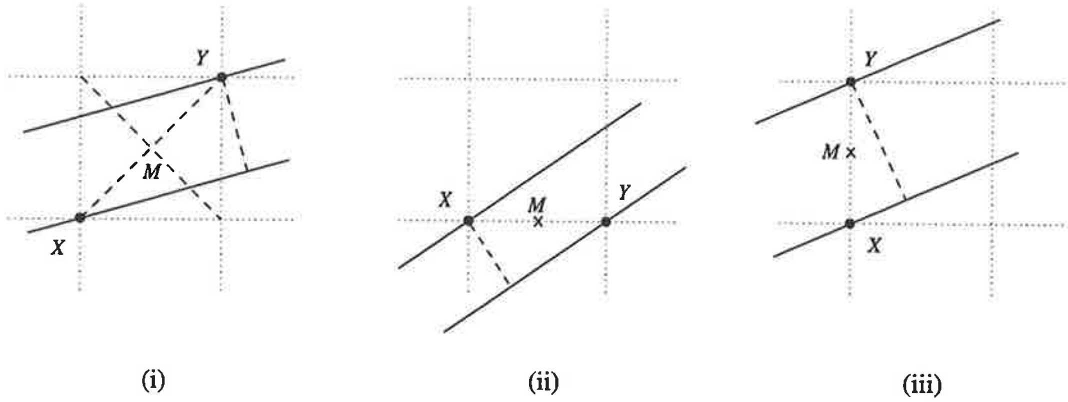


Figure 12.17: The case $L(z_1, z_2)$, $z_1 \geq 2$

In summary, by comparing the results from §12.4, §12.5 and §12.6, we see that

$$\frac{A}{p} \leq \lambda \approx 0.636,$$

with equality when and only K is congruent to the rounded hexagon $\mathcal{U}_2$ of Figure 12.2.

12.7 A conjecture

We now conjecture a stronger form of Theorem 12.5 by removing the symmetry condition on K.

Conjecture 12.7 *Let K be a set in $\mathcal{K}^2$ and let $G(K^o, \Gamma) = 2$. Then*

$$\frac{A}{p} \leq \lambda \approx 0.636.$$

Equality holds when and only when K is congruent to the rounded hexagon $\mathcal{U}_2$ (Figure 12.2).

Chapter 13

Conclusion

In this chapter we outline the scope for further research in the area. Appendix B contains a summary of the results of this thesis. From Tables B.2-B.5 of Appendix B, it may be seen that many new problems remain in the area (indicated with $\star$ in the tables). We now describe possible future directions for the research.

13.1 Problems concerning non-rectangular lattices

In our work, we have been primarily concerned with the rectangular lattice Λ_R, in particular, the integral lattice Γ. New problems arise by considering non-rectangular lattices. For example, Minkowski's Convex Body Theorem as stated in Theorem 1.1 is generalized to the general lattice Λ in Theorem 2.2 by using an affine transformation and observing that $A(K)/\det(\Lambda)$ is an affine-invariant quantity. Wetwitschka (1987) gives an analogue of Minkowski's Convex Body Theorem for the equilateral triangular lattice Λ_T generated by the vectors $(1,0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$. Scott (1978c) and Wetwitschka (1991) find the maximal width of a set K in $\mathcal{K}^2$ having $G(K^o, \Lambda_T) = 0$ and $G(K^o, \Lambda_T) = 1$ respectively. The results here are simple as the lattice is based on the equilateral triangle which turns out to be the extremal figure in these problems. Vassallo (1992) finds the maximal width of a set K in $\mathcal{K}^2$ with $G(K^o, \Lambda) = 0$. Vassallo (1995) also generalizes the area-width result of Chapter 7 and the area-diameter result of Chapter 11 to the general lattice Λ. However, in contrast with Wetwitschka's and Scott's inequalities for Λ_T, Vassallo's results are quite complicated. Vassallo and Wills (1996) also generalize the width-diameter inequality by Scott (1979b) to arbitrary lattices. The area-perimeter inequality obtained by Bender (1962) as stated in Theorem 12.2 is in fact true for the general lattice Λ. Wills (1971) obtains a result for V/S for arbitrary lattices.

Two ideas are useful in generalizing to the lattice Λ. If the quantity under consideration is an affine-invariant quantity, the generalization is a straightforward exercise. The other method is to employ Steiner symmetrization to transform the problem into one concerning rectangular lattices (see for example (Bender 1962; Vassallo 1995)).

13.2 Problems in $\Re^n$

All our problems are posed in the Euclidean plane $\Re^2$. The question arises as to whether these results hold in higher dimensions. As an example, Minkowski's Convex Body Theorem as stated in Theorem 2.2 in fact holds for sets in $\mathcal{K}^n$ (the corresponding result is obtained by replacing the number 4 in Theorem 2.2 by 2^n). Ehrhart (1964) conjectures a result for the maximal volume of a Λ_n-admissible set having O as the centre of gravity. The conjecture has been proved for $n = 2$ (Ehrhart 1955a) and for $n = 3$ for a special class of solids (Ehrhart 1955b). McMullen and Wills (1981) generalize a result by Scott (1973) concerning the maximal width of a set K in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 0$ to sets in $\mathcal{K}^n$. In the same paper an analogue of the width-diameter result in Chapter 6 is given for sets in $\mathcal{K}^n$. Bokowski and Wills (1974) and Schmidt (1972) independently extend the area-perimeter result by Nosarzewska (1948) to the case $n = 3$. In fact, this result holds for all n as proved by Bokowski, Hadwiger and Wills (1972). The area-perimeter inequality obtained by Bender (1962) for sets K in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 0$ has been extended to $n = 3$ and $n = 4$ by Wills (1968, 1970 respectively). The generalization to $\Re^n$ is proved by Hadwiger (1970). From the literature, it is observed that the greatest difficulty in generalizing to $\Re^n$ is encountered in extending the result to $\Re^3$. If this can be done, the generalization to $\Re^n$ is usually a simple induction exercise.

13.3 Problems concerning an arbitrary number of lattice points

In all our problems, we have considered those cases for which $G(K^o, \Gamma) < 3$. What are the corresponding results when K^o contains an arbitrary number of lattice points ? Van der Corput (1935, 1936) gives a relationship between V and $G(K^o, \Lambda_n)$ for an O-symmetric set. Scott (1987) relates A with $G(K^o, \Lambda)$ for the general class of convex sets. Ehrhart (1955c, 1955d) gives partial results relating A and $G(K^o, \Lambda)$ for sets having centre of gravity at O. Nosarzewska (1948) gives an inequality relating A, p and $G(K^o, \Gamma)$. Inequalities concerning w and $G(K^o, \Gamma)$ have been obtained by Elkington and Hammer (1976). Hammer (1964, 1966, 1971, 1979) also obtains inequalities relating A/p, V/S and A/d with $G(K^o, \Gamma_n)$. Reich (1970) obtains a result relating A, p and d with $G(K^o, \Gamma)$. Vassallo (1995) gives a relationship between A/d and $G(K^o, \Lambda)$, while Vassallo and Wills (1996) relate w and d with $G(K^o, \Lambda)$.

13.4 Problems concerning special sets

By taking K to be a set with special properties, we have new and interesting problems. Minkowski's Convex Body Theorem is an example of such a problem, for if the symmetry condition were removed, A will be unbounded. Arkinstall and Scott (1979) find the maximal value of A/p for a Γ-admissible set where O is the centre of symmetry. In an earlier paper, Scott (1978a) conjectures a result for the maximal area of a Λ_n-admissible set in $\mathcal{K}^n$ having its volume equally distributed in the 2^n orthants. The conjecture is confirmed for $n = 2$ in the same paper. Ehrhart (1955a) takes O to be the centre of gravity of a Γ-admissible set and finds the maximal area of the set. Scott (1982) conjectures a result for the maximal width of such a set. In the same paper, Scott also conjectures a result for the maximal area of a Γ-admissible set in $\mathcal{K}^2$ with circumcentre O. We discover that the conjecture is false and have revised the conjecture in §3.5. We also find the maximal circumradius of such a set in Chapter 4. Sallee (1969) finds the maximal width of a set of constant width K in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 0$. Sawyer (1955b) finds $V/\det(\Lambda_n)$ for a set which is symmetric about a point apart from O. In the problems of Chapters 11 and 12, K is taken to be a set containing two interior lattice points symmetrically placed about the centre of the set.

13.5 Problems concerning other geometric functionals

In this thesis we have dealt with the geometric functionals A, p, d, w, r and R on the space $\mathcal{K}^2$. By defining new functionals, many new problems may be investigated. A number of results have been obtained for the inner 1-quermasses, d_i and the outer 1-quermasses, w_i, defined in §6.6. For example, Scott (1985c) obtains an inequality relating V with $w_1, w_2, \ldots, w_n$. Scott (1985b, 1989) also obtains inequalities for d_i. Wills (1990) obtains an inequality relating w_i and d_i for a set in $\mathcal{K}^n$, analogous to the width-diameter inequality in Chapter 6. Another analogue of the width-diameter inequality in Chapter 6 is given by McMullen and Wills (1981) who relate w_i with the functional δ_i defined to be $\max\{w(K \cap H_i)\}$ where H_i is a hyperplane perpendicular to the i-th basis vector. Sawyer (1954) introduces a new functional for a Λ_n-admissible set K as follows. Let

$$\lambda(K) = \lambda = \sup \frac{PO}{OP'},$$

where POP' is a chord of K ($\lambda(K)$ is an example of a coefficient of asymmetry for a set K). Many other similar functionals may be defined (Grünbaum 1963). Sawyer (1954) establishes a relationship between $V/\det(\Lambda_n)$ and λ for Λ_n-admissible sets. He obtains an exact formulation for $A/\det(\Lambda)$ for Λ-admissible sets (Sawyer 1995a). Scott (1974b) defines a functional, λ on K as follows. Let $\lambda(K) = k$ if the set K is $k\Lambda_n$-bounded (A set is $k\Lambda_n$-bounded if some translate of K is contained in a fundamental cell of $k\Lambda_n$ but no translate is contained in any fundamental cell of $(k-\varepsilon)\Lambda_n$ where $\varepsilon > 0$). As expected, there is a relationship between V and λ. Scott finds an exact relationship between A and λ for Λ-admissible sets in $\mathcal{K}^2$.

13.6 Problems involving three parameters

In our work, we have concentrated on inequalities concerning one or two of the geometric parameters A, p, d, w, r and R. What inequalities exist among three of these parameters ? As an example, Reich (1970) proves that if K is a set in $\mathcal{K}^2$ and $G(K^o, \Gamma) = 1$, then $A \leq \frac{1}{2}p + d$. As far as we know, this is the only inequality relating three parameters for lattice constrained sets. Perhaps, as with the one and two parameter problems, the inequalities for unconstrained sets can give some ideas for the corresponding inequalities for lattice constrained sets. We have compiled a list of inequalities relating three parameters for sets with no lattice constraint. This may be found in Appendix C.

It may be seen from the preceding discussion that problems abound in the area. It is hoped that this work will motivate further research in this very fertile area.

Appendix A

Supplementary diagrams for Chapter 3

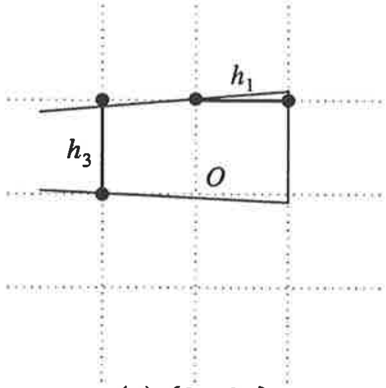
Figure A.1: The unbounded sets for the two intercept case

Figure A.2: The unbounded sets for the three intercept case

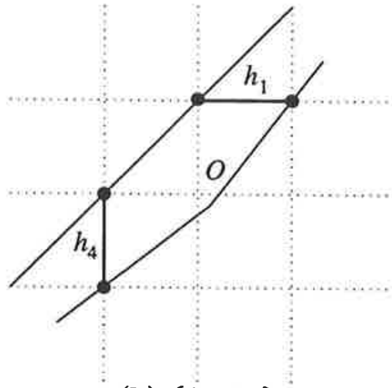
Figure A.3: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_7\}$ case

Figure A.4: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_7\}$ case

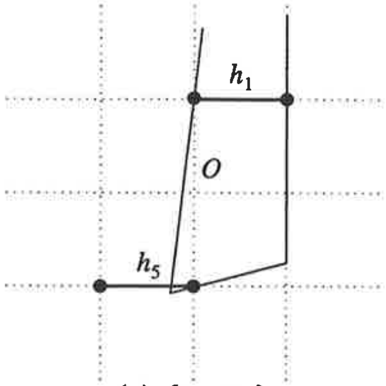
Figure A.5: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_8\}$ case



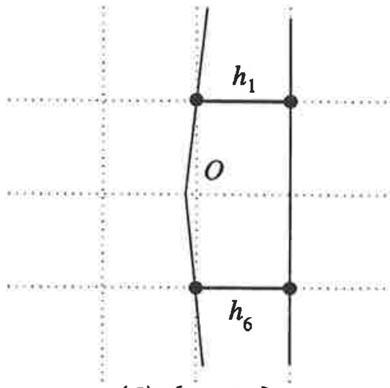
(a) $\{h_1, h_3\}$



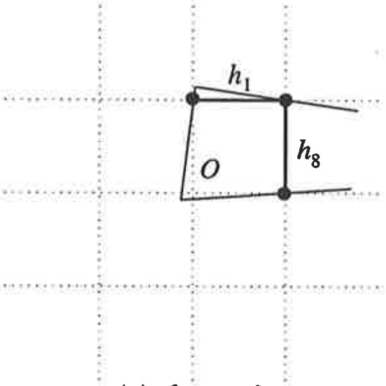
(b) $\{h_1, h_4\}$



(c) $\{h_1, h_5\}$

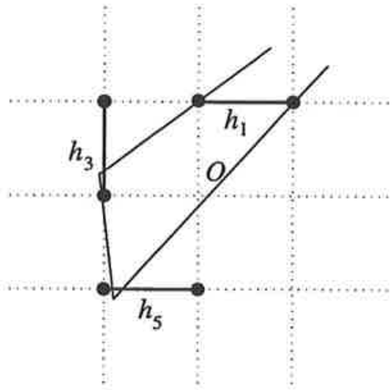


(d) $\{h_1, h_6\}$

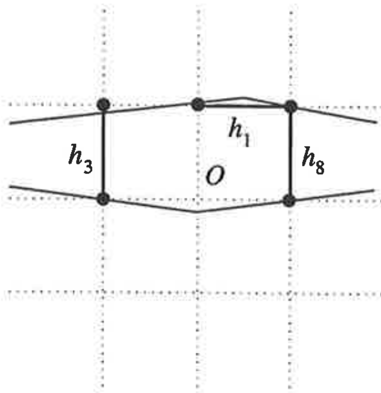


(e) $\{h_1, h_8\}$

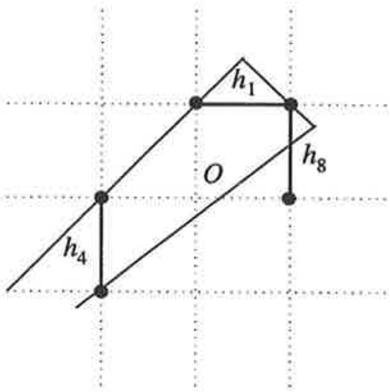
Figure A.1: The unbounded sets for the two intercept case



(a) $\{h_1, h_3, h_5\}$



(b) $\{h_1, h_3, h_8\}$



(d) $\{h_1, h_4, h_8\}$

Figure A.2: The unbounded sets for the three intercept case

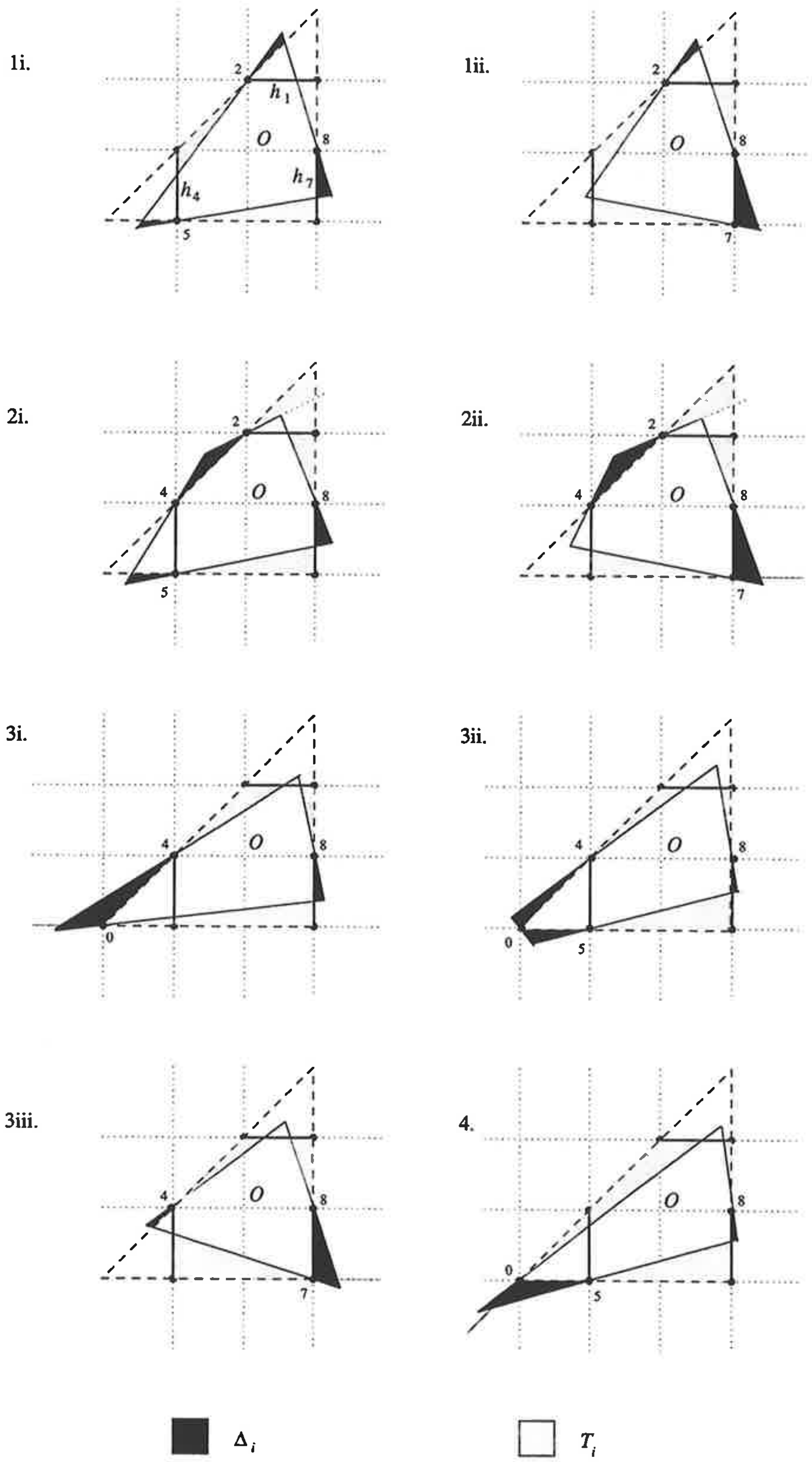


Figure A.3: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_7\}$ case

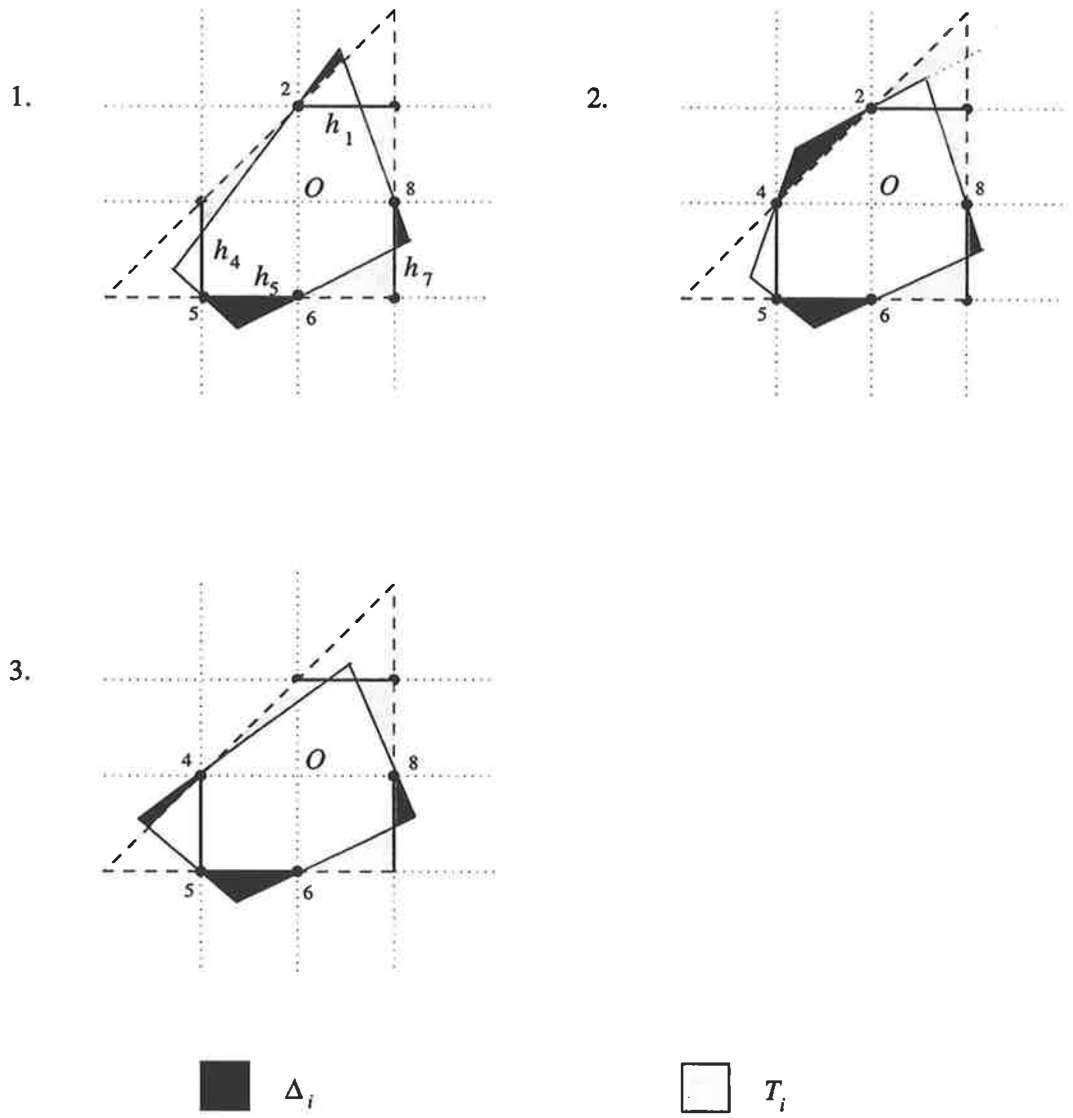


Figure A.4: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_7\}$ case

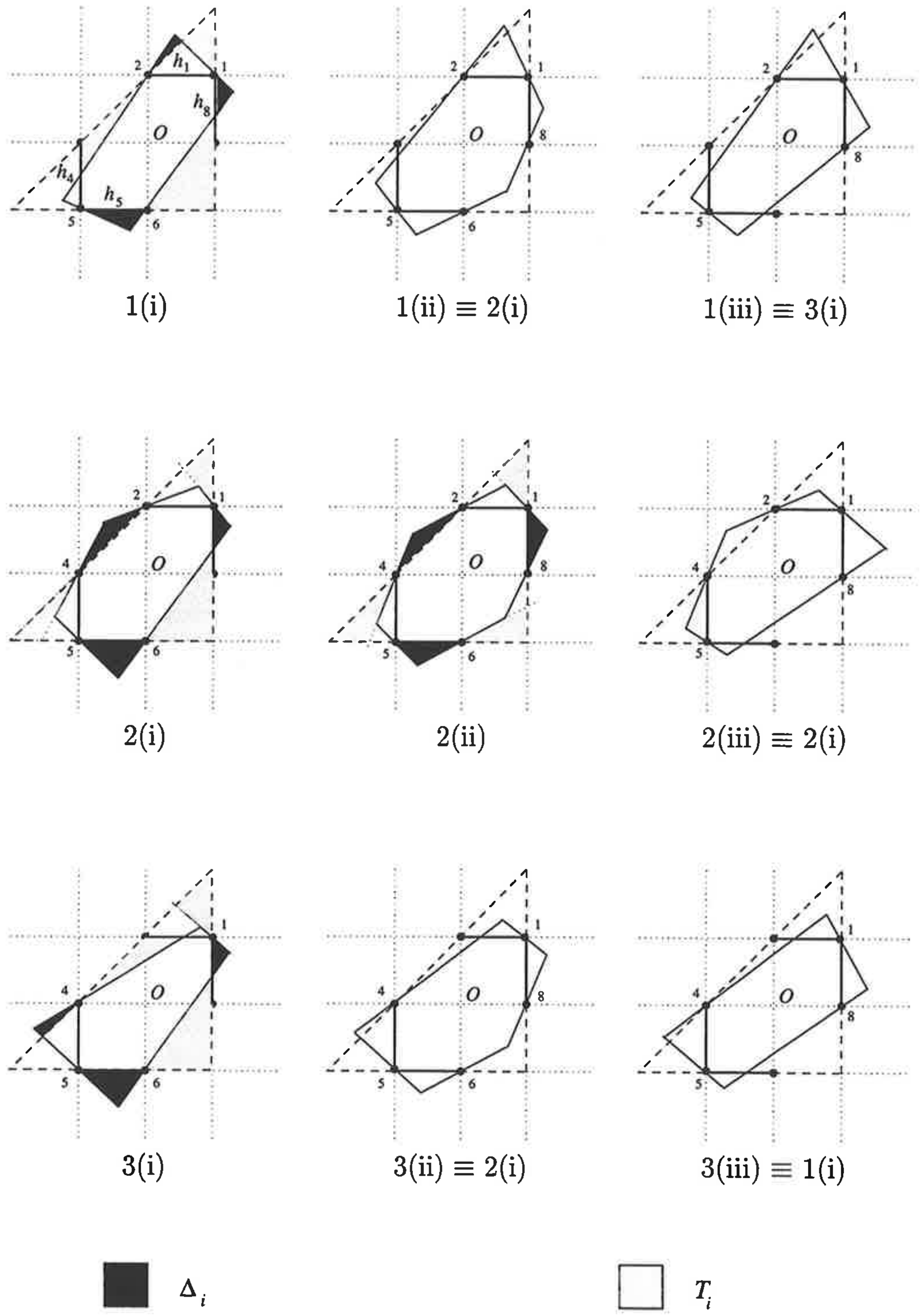


Figure A.5: Triangles $\triangle_i$ and T_i for the $\{h_1, h_4, h_5, h_8\}$ case

Appendix B

Summary of results

This appendix contains a summary of the results of this thesis.

List of Tables

We have indicated with $\star$ those sets of parameters for which the inequalities are not known.

List of Figures

Parameters	Inequality	Extremal Sets	Reference (see note 2)
A, p	$0 < A \leq p^2/(4\pi)$	I, C	p.207, ex.5.8
A, d	$0 < A \leq (\pi d^2)/4$	I, C	p.239, ex.6.10a
A, w	$w^2/\sqrt{3} \leq A < \infty$	E, P	p.221, ex.6.4
A, R	$0 < A \leq \pi R^2$	I, C	†
A, r	$\pi r^2 \leq A < \infty$	C, P	†
p, d	$2d < p \leq \pi d$	I, W	p.257, ex.7.17a
p, w	$\pi w \leq p < \infty$	W, P	p.258, ex.7.18a
p, R	$4R < p \leq 2\pi R$	I, W	†
p, r	$2\pi r \leq p < \infty$	C, P	†
d, w	$w \leq d < \infty$	W, P	†
d, R	$\sqrt{3}R \leq d \leq 2R$	E, C	p.213, ex.6.1 (see note 4)
d, r	$2r \leq d < \infty$	C, P	†
w, R	$0 < w \leq 2R$	I, C	†
w, r	$2r \leq w \leq 3r$	C, E	p.215, ex.6.2 (see note 5)
R, r	$r \leq R < \infty$	C, P	†

Table B.1: Inequalities for sets with no lattice constraint

Notes on Table B.1:

1. Notation:
   - I line segment
   - C circle
   - E equilateral triangle
   - W orbiforms (sets of constant width)
   - P parallel strip

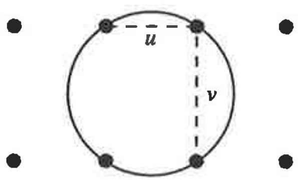
2. The proofs of the results in the table may be found in the indicated pages of the book by Yaglom and Boltyanskii (1961). Results indicated with † are trivial and their proofs have been omitted.
3. The left hand and right hand extremal sets correspond to the left hand and right hand inequalities respectively.
4. The left hand inequality for the pair (d, R) is referred to as Jung's Theorem.
5. The right hand inequality for the pair (w, r) is referred to as Blaschke's Theorem.
6. Where an inequality is strict, the extremal set shows the bound to be best possible.

Parameters	Inequality	Extremal Sets	Reference
A	unbounded		
p	unbounded		
d	unbounded		
w	$w \leq \frac{1}{2}(2v + \sqrt{3}u)$	$\mathcal{E}_R$	(Scott 1993), (Vassallo 1992), Chapter 5
R	unbounded		
r	$r \leq \frac{1}{2}\sqrt{u^2 + v^2}$	$\mathcal{C}_R$	(see note 2)
A, p	$A/p < \frac{1}{2}v$	$\mathcal{P}_R$	(Bender 1962)
A, d	$A/d \leq \lambda$, $\lambda = \max\{v, 2\sin(\phi/2)\sqrt{u^2 + v^2}\}$, $\phi \approx 0.832 \approx 47.7^o$	$\mathcal{P}_R$ or $\mathcal{H}_R$ (see note 3)	(Vassallo 1995)
A, w	1. $(w - v)A \leq \frac{1}{2}uw^2$ 2. $\frac{A}{w^3} \leq \frac{1}{\sqrt{3}}(v + \frac{\sqrt{3}}{2}u)^{-1}$	$\mathcal{T}_R$ $\mathcal{E}_R$	Chapter 7 Chapter 8
A, R	$A/R < 2\lambda$	$\mathcal{P}_R$ or $\mathcal{H}_R$	(see note 3)
A, r	$\star$		
p, d	$\star$		
p, w	$(w - v)p \leq 3uw$	$\mathcal{E}_R$	Chapter 7
p, R	$\star$		
p, r	$\star$		
d, w	$(w - v)(d - u) \leq uv$	$\mathcal{T}_R$	Chapter 6
d, R	$2R - d \leq \frac{2}{3}(2 - \sqrt{3})(v + \frac{\sqrt{3}}{2}u)$	$\mathcal{E}_R$	Chapter 9
d, r	$(2r - v)(d - u) < uv$	$\mathcal{P}_R$	Chapter 10
w, R	$(w - v)R \leq \frac{1}{\sqrt{3}}uw$	$\mathcal{E}_R$	Chapter 7
w, r	$w - 2r \leq \frac{1}{3}(v + \frac{\sqrt{3}}{2}u)$	$\mathcal{E}_R$	Chapter 9
R, r	$\star$		

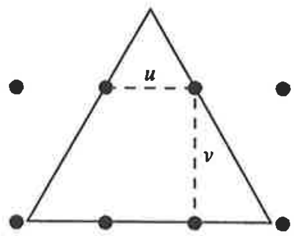
Table B.2: Inequalities for $G(K^o, \Lambda_R) = 0$

<u>Notes on Table B.2</u>:

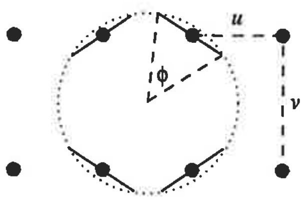
1. The lattice Λ_R is generated by $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$, where $u \leq v$.
2. The result for r has not been proved in the thesis. However it follows easily by symmetrizing K about the lines $x = u/2$ and $y = v/2$ and noting from Theorem 2.3 that symmetrization does not decrease the inradius.
3. For the pairs (A, d) and (A, R), the extremal set is $\mathcal{P}_R$ if $\lambda = v$ and $\mathcal{H}_R$ otherwise. The result for (A, R) follows easily from $d \leq 2R$ and the (A, d) inequality.
4. Where an inequality is strict, the extremal set shows the bound to be best possible.



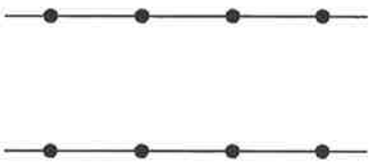
The circle $\mathcal{C}_R$



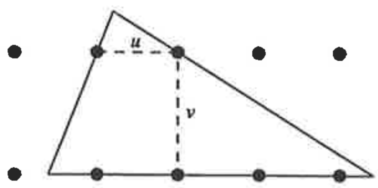
The equilateral triangle $\mathcal{E}_R$



The truncated rectangle $\mathcal{H}_R$, $\phi \approx 47.7^o$



The parallel strip $\mathcal{P}_R$



The triangle $\mathcal{T}_R$

Figure B.1: Extremal sets for the case $G(K^o, \Lambda_R) = 0$

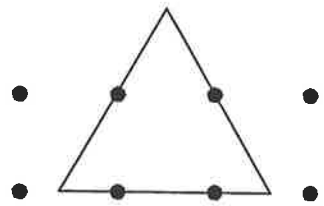
Parameters	Inequality	Extremal Sets	Reference
A	unbounded		
p	unbounded		
d	unbounded		
w	$w \leq \frac{1}{2}(2+\sqrt{3}) \approx 1.866$	$\mathcal{E}_0$	(Scott 1973)
R	unbounded		
r	$r \leq 1/\sqrt{2}$	$\mathcal{C}_0$	(see note 1)
A, p	$A < \frac{1}{2}p$	$\mathcal{P}_0$	(Bender 1962)
A, d	$A/d \leq \lambda, \lambda \approx 1.144$	$\mathcal{H}_0$	(Scott 1974a)
A, w	1. $(w-1)A \leq \frac{1}{2}w^2$ 2. $\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(1+\frac{\sqrt{3}}{2})^{-1} \approx 0.309$	$\mathcal{T}_0$ $\mathcal{E}_0$	(Scott 1980) Chapter 8
A, R	$A/R \leq 2\lambda,\ \lambda \approx 1.144$	$\mathcal{H}_0$	Chapter 11
A, r	1. $(2r-1)A \leq 2(\sqrt{2}-1) \approx 0.828$ 2. $(2r-1)\lvert A-1\rvert < 1/2$	$\mathcal{S}_0$ $\mathcal{P}_0$	Chapter 10 Chapter 10
p, d	$\star$		
p, w	$(w-1)p \leq 3w$	$\mathcal{E}_0$	(Scott 1980)
p, R	$\star$		
p, r	1. $(2r-1)p \leq \frac{4}{r}(\sqrt{2}-1)$ 2. $(2r-1)\lvert p-4\rvert < 2$	$\mathcal{S}_0$ $\mathcal{P}_0$	Chapter 10 Chapter 10
d, w	$(w-1)(d-1) \leq 1$	$\mathcal{T}_0$	(Scott 1978b)
d, R	$2R - d \leq \frac{1}{3}$	$\mathcal{E}_0$	Chapter 9
d, r	$(2r-1)(d-1) < 1$	$\mathcal{P}_0$	Chapter 10
w, R	$(w-1)R \leq \frac{1}{\sqrt{3}}w$	$\mathcal{E}_0$	(Scott 1980)
w, r	$w - 2r \leq \frac{1}{3} + \frac{1}{6}\sqrt{3} \approx 0.622$	$\mathcal{E}_0$	Chapter 9
R, r	$\star$		

Table B.3: Inequalities for $G(K^o, \Gamma) = 0$

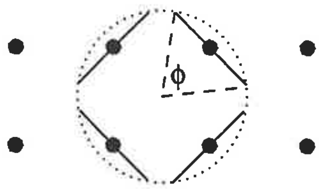
Notes on Table B.3:

1. The result for r follows easily from the corresponding result in Table B.2 by letting $u = v = 1$. In fact, all the above inequalities with the exception of the (A, r) and (p, r) inequalities follow immediately by letting $u = v = 1$.
2. The methods used to prove the (A, r) and (p, r) inequalities may be extended to obtain the corresponding inequalities for the rectangular lattice. However the results are complicated and have been omitted in Table B.2.
3. Where an inequality is strict, the extremal set shows the bound to be best possible.

The circle $\mathcal{C}_0$

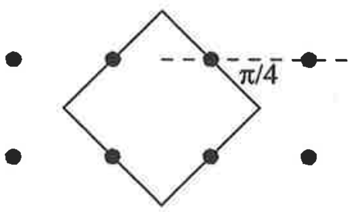


The equilateral triangle $\mathcal{E}_0$

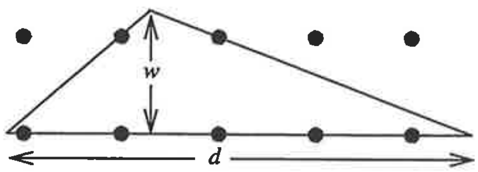


The truncated diagonal square $\mathcal{H}_0$, $\phi \approx 47.7^o$

The parallel strip $\mathcal{P}_0$



The diagonal square $\mathcal{S}_0$



The triangle $\mathcal{T}_0$

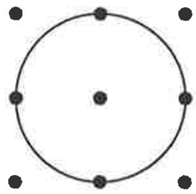
Figure B.2: Extremal sets for the case $G(K^o, \Gamma) = 0$

Parameters	Inequality	Extremal Sets	Reference
A	1. $A \leq 4$ if O is centre of K 2. $A \leq 4.5$ if O is the C.G. 3. $A < 4.5$ or A unbounded 4. *Conjecture:* If O is the circumcentre then $A \leq \alpha \approx 4.04569$	e.g. $\mathcal{S}_1$ Ehrhart's $\triangle$ Ehrhart's $\triangle$ $\mathcal{Q}_1$	(Minkowski 1911) (Ehrhart 1955a) Chapter 3 Chapter 3
p	unbounded		
d	unbounded		
w	1. $w \leq 1+\sqrt{2} \approx 2.414$ 2. *Conjecture:* If O is the C.G. then $w \leq 3\sqrt{2}/2$	$\mathcal{I}_1$ Ehrhart's $\triangle$	(Scott 1985a) (Scott 1982)
R	$R \leq \alpha \approx 1.685$ or R unbounded	$\mathcal{T}$	Chapter 4
r	$r \leq 1$	$\mathcal{C}_1$	(see note 1)
A, p	$A/p \leq 2(2+\sqrt{\pi})^{-1} \approx 0.53$ (O is centre of K)	$\mathcal{U}_1$	(Arkinstall and Scott 1979)
A, d	$A/d \leq \sqrt{2}\lambda d, \lambda \approx 1.144$	$\mathcal{H}_1$	Chapter 11
A, w	1. $A(w-\sqrt{2}) \leq \frac{1}{\sqrt{2}}w^2$ 2. *Conjecture:* $\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}\cdot\frac{4}{\sqrt{2(5+\sqrt{3})}} \approx 0.243$	$\mathcal{T}_1$ $\mathcal{E}_1$	Chapter 7 Chapter 8
A, R	$A/R \leq 2\sqrt{2}\lambda R$	$\mathcal{H}_1$	Chapter 11
A, r	$A(2r-\sqrt{2}) \leq 4(2-\sqrt{2}) \approx 2.343$	$\mathcal{S}_1$	Chapter 10
p, d	$\star$		
p, w	$\star$		
p, R	$\star$		
p, r	$p(2r-\sqrt{2}) \leq \frac{8}{r}(2-\sqrt{2})$	$\mathcal{S}_1$	Chapter 10
d, w	$(w-\sqrt{2})(d-\sqrt{2}) \leq 2$	$\mathcal{T}_1$	(Scott 1985a)
d, R	*Conjecture:* $2R-d \leq \frac{\sqrt{2}}{6}.(7-3\sqrt{3}) \approx 0.425$	$\mathcal{E}_1$	Chapter 9
d, r	$\star$		
w, R	$\star$		
w, r	*Conjecture:* $w-2r \leq \frac{\sqrt{2}}{12}(5+\sqrt{3}) \approx 0.793$	$\mathcal{E}_1$	Chapter 9
R, r	$\star$		

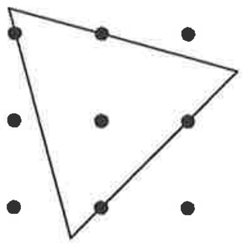
Table B.4: Inequalities for $G(K^o, \Gamma) = 1$

Notes on Table B.4:

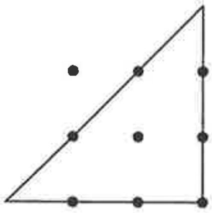
1. The result for r follows easily from the corresponding result in Table B.2 by using the sublattice argument $SL1$. In fact the inequalities for the pairs (A, d), (A, w), (A, R), (p, r) and (w, d) are obtained using the same argument.



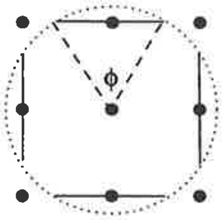
The circle $\mathcal{C}_1$



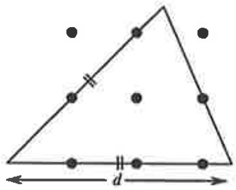
The equilateral triangle $\mathcal{E}_1$



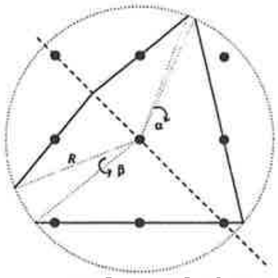
Ehrhart's $\triangle$



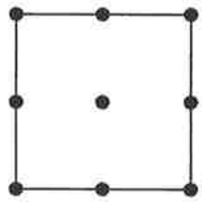
The truncated square $\mathcal{H}_1$, $\phi \approx 47.7^o$



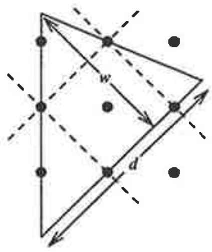
The isosceles triangle $\mathcal{I}_1$



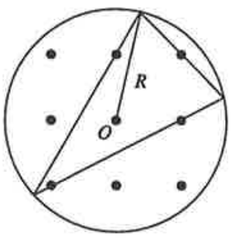
The truncated quadrilateral $\mathcal{Q}_1$,
$R \approx 1.593$, $\alpha \approx 5.47^o$, $\beta \approx 20.23^o$



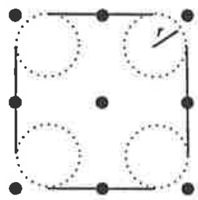
The square $\mathcal{S}_1$



The triangle $\mathcal{T}_1$



The triangle $\mathcal{T}$,
$R \approx 1.685$



The rounded square $\mathcal{U}_1$,
$r \approx 0.530$

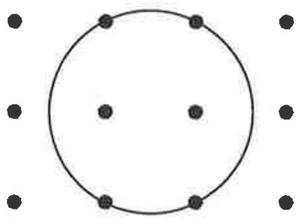
Figure B.3: Extremal sets for the case $G(K^o, \Gamma) = 1$

Parameters	Inequality	Extremal Sets	Reference
A	unbounded		
p	unbounded		
d	unbounded		
w	$w \le \frac{1}{2}(4+\sqrt{3}) \approx 2.866$	$\mathcal{E}_2$	Chapter 5
R	unbounded		
r	$r \le \frac{1}{2}\sqrt{5}$	$\mathcal{C}_2$	(see note 1)
A, p	1. $A/p \le \lambda \approx 0.636$ (special conditions apply, see note 2) 2. *Conjecture:* $A/p \le \lambda \approx 0.636$	$\mathcal{U}_2$ $\mathcal{U}_2$	Chapter 12 Chapter 12
A, d	1. $A/d \le \alpha \approx 1.841$ (special conditions apply, see note 2) 2. *Conjecture:* $A/d \le \alpha \approx 1.841$	$\mathcal{H}_2$ $\mathcal{H}_2$	Chapter 11 Chapter 11
A, w	1. $(w-2)A \le \frac{1}{2}w^2$ 2. $\frac{A}{w^3} \ge \frac{1}{\sqrt{3}}(2+\frac{\sqrt{3}}{2})^{-1} \approx 0.201$	$\mathcal{T}_2$ $\mathcal{E}_2$	Chapter 7 Chapter 8
A, R	1. $A/R \le 2\alpha \approx 3.682$ (special conditions apply, see note 2) 2. *Conjecture:*$A/R \le 2\alpha \approx 3.682$	$\mathcal{H}_2$ $\mathcal{H}_2$	Chapter 11 Chapter 11
A, r	$\star$		
p, d	$\star$		
p, w	$(w-3)p \le 3w$	$\mathcal{E}_2$	Chapter 7
p, R	$\star$		
p, r	$\star$		
d, w	$(w-2)(d-1) \le 2$	$\mathcal{T}_2$	Chapter 6
d, R	$2R - d \le \frac{1}{3}(5-2\sqrt{3}) \approx 0.512$	$\mathcal{E}_2$	Chapter 9
d, r	$\star$		
w, R	$(w-2)R \le \frac{1}{\sqrt{3}}w$	$\mathcal{E}_2$	Chapter 7
w, r	$w - 2r \le \frac{1}{3}(2+\frac{1}{2}\sqrt{3}) \approx 0.955$	$\mathcal{E}_2$	Chapter 9
R, r	$\star$		

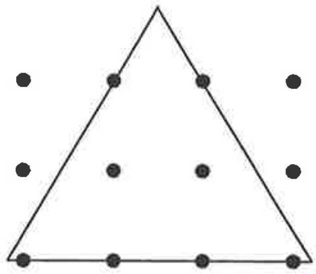
Table B.5: Inequalities for $G(K^o, \Gamma) = 2$

Notes on Table B.5:

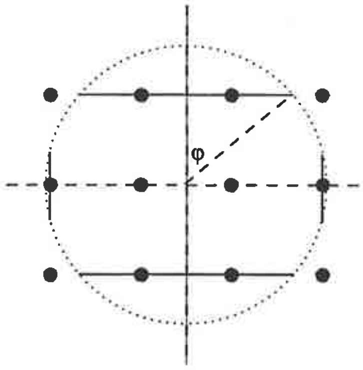
1. The result for r follows easily from the corresponding result in Table B.2 by using the sublattice argument $SL2$. In fact all the above inequalities except those for the pairs (A, p) and (A, d) and (A, R) are obtained by the same argument.
2. The results concerning the pairs (A, p), (A, d) and (A, R) apply to the class of symmetric sets in $\mathcal{K}^2$, having the interior lattice points symmetrically placed about the centre of the set. The conjectures apply to the general convex set containing two interior lattice points.



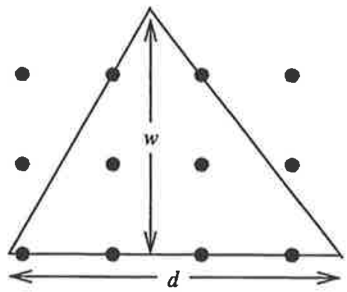
The circle $\mathcal{C}_2$



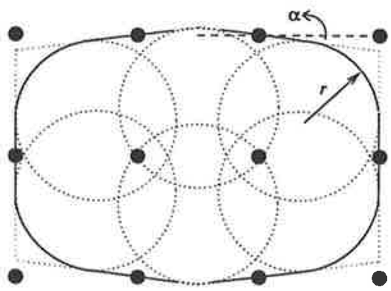
The equilateral triangle $\mathcal{E}_2$



The truncated rectangle $\mathcal{H}_2$, $\varphi \approx 48.5^o$



The triangle $\mathcal{T}_2$



The rounded hexagon $\mathcal{U}_2$,
$\alpha \approx 6.13^o$, $r \approx 0.636$

Figure B.4: Extremal sets for the case $G(K^o, \Gamma) = 2$

Appendix C

Three parameter relationships for sets with no lattice constraint

Let K be a non-empty set in $\mathcal{K}^2$. We compile a list of inequalities relating any three of the geometric functionals A, p, d, w, r and R. We list the details in the following order: the inequality; a set for which the inequality is best (where the inequality is strict, the indicated set shows that the bound is best possible); references for a proof of the inequality.

1. Parameters: A, p, d.

(i) $8\phi A \le p(p - 2d\cos\phi)$ where $2\phi d = p\sin\phi$; the intersection of two disks of equal radius (Figure 13.5a); (Kubota 1923; Yaglom and Boltyanskii 1961, p.240, ex.6.11a).
(ii) $\frac{1}{4}d(p-2d) \le A \le \frac{1}{4}pd$; the right hand inequality is best for a circle; (Hayashi 1923).
(iii) $A > \frac{1}{2}d(p-2d)$; infinite isosceles triangle; (Favard 1929).
(iv) If $2d \le p \le 3d$, then $A \ge \frac{1}{4}(p-2d)(4pd-p^2)^{1/2}$; isosceles triangle; (Kubota 1923, 1924; Yaglom and Boltyanskii 1961, p.229, ex.6.8a).
(v) If $3d \le p \le \pi d$, then $A \ge \frac{1}{4}\sqrt{3}d(p-2d)$; Not best possible unless $p = 3d$ in which case we have the equilateral triangle; (Kubota 1924).

2. Parameters: A, p, w.

(i) $A \le \frac{1}{2}w(p - \frac{1}{2}\pi w)$; convex hull of two disks of equal radius (Figure 13.5d); (Kubota 1923; Yaglom and Boltyanskii 1961, p.241, ex.6.11b).
(ii) If $0 \le 2\sqrt{3}w \le p$, then $A \ge A_1^*$ where A_1^* is the middle root of the equation $128px^3 - 16w(5p^2+w^2)x^2 + 16w^2p^3x - w^3p^4 = 0$; isosceles triangle; (Yamanouti 1932; Yaglom and Boltyanskii 1961, p.231, ex.6.8b).
(iii) If $0 < \pi w < p < 2\sqrt{3}w$, then $A \ge \frac{1}{2}w(p - \sqrt{3}w\sec^2\gamma)$ where $\tan\gamma - \gamma = \frac{1}{6}(p - \pi w)/w$; equilateral Yamanouti triarc (Figure 13.5b); (Kubota and Hemmi 1953; Sholander 1952).
(iv) If $0 < \pi w = p$, then $A \ge \frac{1}{2}(\pi - \sqrt{3})w^2$; Reuleaux triangle (Figure 13.5c); (Lebesgue 1914, 1921; Yaglom and Boltyanskii 1961, p.260, ex.7.20).
(v) $A \ge \frac{1}{6}(4\sqrt{3}w^2 - pw)$; equilateral triangle; (Sholander 1952).
(vi) $A \ge \frac{1}{4}(pw - \frac{2}{\sqrt{3}}w^2)$; equilateral triangle; (Kawai 1932).

(vii) $A \geq \frac{1}{6}pw$; equilateral triangle; (Sholander 1952).

3. Parameters: A, p, r.

(i) $A \leq r(p - \pi r)$; circle; (Bonnesen 1929).
(ii) $A \geq \frac{1}{2}pr$; triangle; (Bonnesen and Fenchel 1934).

4. Parameters: A, p, R.

(i) $A \leq R(p - \pi R)$; circle; (Bonnesen 1929).
(ii) $A < 2R(p - 2R)$; not best possible; (Henk and Tsintsifas 1994).
(iii) $A > R(p - 4R)$; infinite isosceles triangle; (Favard 1929).

5. Parameters: A, d, w.

(i) $A < wd$; rectangle with width tending towards 0; (Kubota 1923).
(ii) $A \leq \frac{1}{2}w(d^2 - w^2)^{1/2} + d^2 \sin^{-1}(w/d)$; a set formed by removing from a disk points outside two symmetrically placed chords (Figure 13.5e); (Kubota 1923; Yaglom and Boltyanskii 1961, p.240, 6.10b).
(iii) If $0 \leq w \leq (\sqrt{3}/2)d$, then $A \geq \frac{1}{2}wd$; triangle; (Kubota 1923; Yaglom and Boltyanskii 1961, p.227, ex.6.7).
(iv) If $\sqrt{3}/2 < w < d$, then $A \geq \frac{1}{2}(\pi w^2 - \sqrt{3}d^2) + 3w^2(\tan\delta - \delta)$, where $\delta = \cos^{-1}(w/d)$; equilateral Yamanouti triarc (Figure 13.5b); (Kubota and Hemmi 1953; Sholander 1952).
(v) If $w = d$, then $A \geq \frac{1}{2}(\pi - \sqrt{3})d^2$; Reuleaux triangle (Figure 13.5c); (Lebesgue 1914, 1921; Yaglom and Boltyanskii 1961, p.260, ex.7.20).
(vi) $A \geq \frac{1}{2}(3dw - \sqrt{3}d^2)$; equilateral triangle; (Sholander 1952).
(vii) $A \geq \frac{1}{2}(\pi w^2 - \sqrt{3}d^2)$; Reuleaux triangle (Figure 13.5c); (Sholander 1952).

6. Parameters: A, d, r.

(i) $A < 2dr$; parallel strip; (Henk and Tsintsifas 1994).

7. Parameters: A, d, R.

(i) $(2R - d)A \leq \pi(3\sqrt{3} - 5)R^3$; not best possible; (Scott 1981).
(ii) *Conjecture*: $(2R - d)A \leq \frac{3}{2}(2 - \sqrt{3})(\pi - \sqrt{3})R^3$, with equality for a Reuleaux triangle (Figure 13.5c); (Scott 1981).

8. Parameters: A, w, r.

(i) $(w - 2r)A < \frac{1}{4}w^3$; an infinite isosceles triangle; (Scott 1979a).
(ii) $(w - 2r)A \leq \frac{1}{\sqrt{3}}w^2 r$; equilateral triangle; (Scott 1979a).
(iii) $(w - 2r)A \leq \sqrt{3}wr^2 \leq 3\sqrt{3}r^3$; equilateral triangle; (Scott 1979a).

9. Parameters: A, w, R.

(i) $A < 2wR$; rectangle with width tending towards 0, (Henk and Tsintsifas 1994).

(ii) $A \geq \frac{\sqrt{3}}{2}wR$; equilateral triangle; (Henk 1991).

10. Parameters: A, r, R.

(i) $A < 4Rr$; parallel strip; (Henk and Tsintsifas 1994).
(ii) $A \geq 2Rr$; a triangle with width tending towards 0; (Henk and Tsintsifas 1994).

11. Parameters: p, d, w.

(i) $p \leq 2(d^2 - w^2)^{1/2} + 2d\sin^{-1}(w/d)$; a set formed by removing from a disk points outside two symmetrically placed chords (Figure 13.5e); (Kubota 1923; Yaglom and Boltyanskii 1961, p.257, ex.7.17b).
(ii) $p \geq 2(d^2 - w^2)^{1/2} + 2w\sin^{-1}(w/d)$; the convex hull of a disk and two symmetrically placed points (Figure 13.5f); (Kubota 1923; Yaglom and Boltyanskii 1961, p.258, ex.7.18b).

12. Parameters: p, d, r.

(i) $p < 2d + 4r$; a rectangle with width tending towards 0; (Henk and Tsintsifas 1994).

13. Parameters: p, d, R.

(i) $(2R - d)p \leq (2\sqrt{3} - 3)\pi R^2$; sets of constant width; (Scott 1981).

14. Parameters: p, w, r.

(i)$(w - 2r)p \leq \frac{2}{\sqrt{3}}w^2$; equilateral triangle; (Scott 1979a).
(ii) $(w - 2r)p \leq 2\sqrt{3}wr \leq 6\sqrt{3}r^2$; equilateral triangle; (Scott 1979a).

15. Parameters: p, w, R.

We have not been able to find any result relating these parameters.

16. Parameters: p, r, R.

We have not been able to find any result relating these parameters.

17. Parameters: d, w, r.

(i) $(w - 2r)d \leq 2\sqrt{3}r^2$, equilateral triangle; (Scott 1978b, 1979a).
(ii) $(w - 2r)d < \frac{1}{2}w^2$; the infinite isosceles triangle; (Scott 1979a).
(iii) $(w - 2r)d \leq \frac{2}{\sqrt{3}}wr$; equilateral triangle; (Scott 1979a).

18. Parameters: d, w, R.

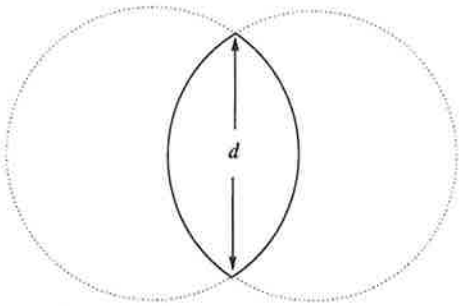
(i) $(2R - d)w \leq \sqrt{3}(2 - \sqrt{3})R^2$; Reuleaux triangle (Figure 13.5c); (Scott 1981).
(ii) $(2R - d) \leq \frac{2}{3}(2 - \sqrt{3})w$; equilateral triangle; Chapter 9 of this thesis.

19. Parameters: d, r, R.

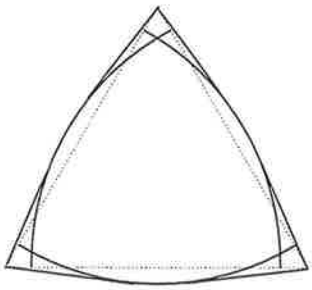
(i) $(2R - d)r \leq (3\sqrt{3} - 5)R^2$; Reuleaux triangle (Figure 13.5c); (Scott 1981).

20. Parameters: w, r, R.

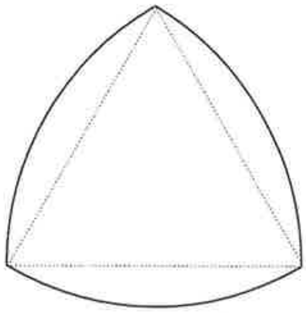
(i) $(w - 2r)R < \frac{1}{4}w^2$; infinite isosceles triangle; (Scott 1979a).
(ii) $(w - 2r)R \leq \frac{2}{3}wr$; equilateral triangle; (Scott 1979a).
(iii) $(w - 2r)R \leq 2r^2$; equilateral triangle; (Scott 1979a).



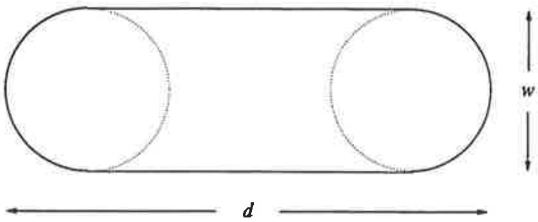
(a) Intersection of two circles



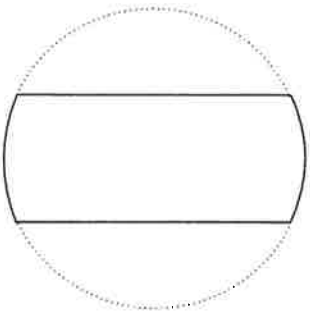
(b) Equilateral Yamanouti triarc



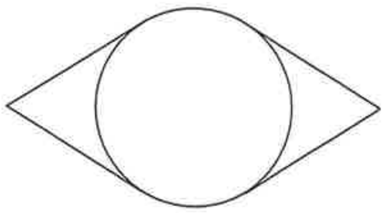
(c) Reuleaux triangle



(d) Convex hull of two circles



(e) Truncated circle



(f) Convex hull of circle and two points

Figure C.1: Extremal sets for three parameter problems

References

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