THE UNIVERSITY OF ADELAIDE

Modelling and Identification of Dynamic Systems using Modal and Spectral Data

by

Senad A. Burak

The University of Adelaide
Faculty of Engineering
Department of Mechanical Engineering
South Australia 5005

January 1997

A thesis submitted in fulfilment of the requirements for the degree of

Doctor of Philosophy

TABLE OF CONTENTS

LIST OF SYMBOLS

$a_{(e)}$, $b_{(e)}$	physical parameters of element (e)
A, $A(x)$	cross sectional area
a, **b**, **d**	column vectors with
$\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$, **E**	square matrices of order n
$a_i, b_i, p_i, q_i, a_i^*, b_i^*, p_i^*, q_i^*$	coefficients
ε	tolerance
E, $E(x)$	the Young's modulus of elasticity
$f(x)$	continuous function
$\mathbf{G}_{(e)}$, $\mathbf{Y}_{(e)}$	element square matrices
h	length of the segment
I	unit matrix of order n
I, $I(x)$	moment of inertia
K	stiffness matrix
k_i	stiffness of the segment i
k_{ij}^H, k_{ij}^V, k_{ij}^D, k_{ij}^Z	physical parameters
L	length of the physical system
λ, λ_i, μ, μ_i	eigenvalue
λ_{mn}	natural frequencies of a two-dimensional system
M	mass matrix
m_i	mass of the segment i
n, N	number of degrees of freedom
Ne	number of elements
$p(x)$, p_i	physical parameters
P, Q	forces
ρ, $\rho(x)$	mass per unit length
u, $\mathbf{u}^{(i)}$	eigenvectors
$w(x,t)$	axial displacement
ω, ω_i	natural frequencies

ABSTRACT

Modelling and Identification of Dynamic Systems using Modal and Spectral Data

Abstract

A frequently encountered engineering problem is to determine the physical parameters of vibrating systems from the knowledge of some of their dynamic response characteristics, ie. the natural frequencies (eigenvalues) and mode shapes (eigenvectors). Unlike the inverse eigenvalue approach, which is based on the knowledge of the eigenvalues of the model, we use in this thesis both, eigenvalues and eigenvectors. We show that systems with known connectivity, ie. the finite difference or finite element models, may be reconstructed if small number of eigenvalues and their associated eigenvectors are known. Since these data may be accurately determined from the frequency response function obtained by experimental modal analysis, the proposed numerical algorithms may be effectively used as a practical tool for the solution of the problem.

In the first part of the thesis we used this approach to reconstruct the physical parameters of an axially vibrating rod, modelled using the four-point finite difference scheme. Next, we gave a methodology for the solution of the problem that is valid

for a general discrete model of one-dimensional vibrating systems, while in Chapter 5. we solve the problem for some multi-dimensional models.

In order to validate the practical applicability and behaviour of the proposed methodology, an appropriate experiment is carries out. A model of multi-storey building, which may be accurately modelled as a discrete mass-spring system, is chosen for the testing. The dynamic characteristics of the model, ie. the natural frequencies and their corresponding mode shapes are determined from the frequency response function obtained by an experimental modal analysis equipment. Using certain two eigenpairs corresponding to these data we applied the reconstruction algorithm developed in chapter 3 and determined the mass and stiffness parameters of the model. It is shown that the algorithm produce excellent results using certain eigendata, which indicates a practical use of the proposed methodology.

STATEMENT OF ORIGINALITY

To the best of my knowledge and belief all of the material presented in this thesis, except where otherwise referenced, is my own original work, and has not been presented previously for the award of any other degree or diploma in any University. If accepted for the award of the degree of Doctor of Philosophy, I consent that this thesis be made available for loan and photocopying.

Signed: Date: 29/05/97

(Senad A. Burak)

ACKNOWLEDGMENTS

I would like to acknowledge the effort of colleagues and staff in the Mechanical Engineering Department of The University of Adelaide who have made possible to do this research and helped in some way during my studies.

In particular, I wish to express sincere appreciation to Dr. Yitshak M. Ram for his supervision, guidance and encouragement in the research and preparation of this thesis.

I am also grateful for the assistance and encouragement given by Dr. M. A. Wahab and Prof. R. E. Luxton. Special thanks to D. Sivan and B. Cazzolato for their assistance in the experimental part of the work and helpful comments and advice.

CHAPTER 1.

Introduction

> "*A problem is called an inverse problem because of its relationship to another problem, called a direct problem, and because the unknowns in the former are the data, or known, in the latter, and vice versa*"[1].

The objective of this work is to study some new *inverse problems* related to mechanical systems, which are typical to the theory of vibration and engineering practice. The first question associated with the term inverse problem is "*inverse relative to what*?" To answer this question we consider the classical problems, now called *direct problems*. The aim of the direct problems in vibration is to predict the response of a system from the knowledge of its physical characteristics, the governing laws of motion and the external forces. This may be presented in the following diagram:

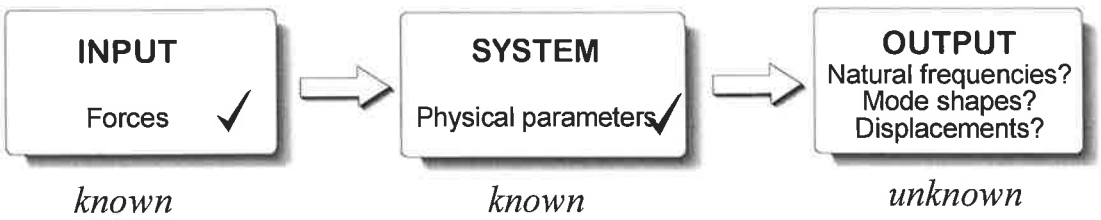

Figure 1-1 A direct problem in vibration

[1] G. M. L. Gladwell and B. R. Zhu [35]

At this stage, the theory of these problems is well established and known. Inverse problems in vibration involve the determination of some physical parameters of a system based on the knowledge of its dynamic characteristics, ie. natural frequencies and mode shapes. A block diagram representation of a typical inverse problem in vibration is shown below:

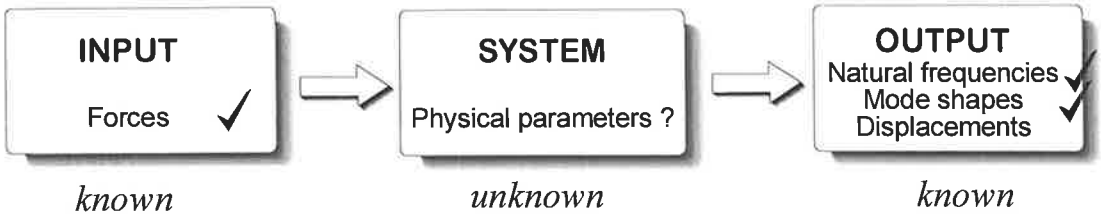

Figure 1-2 An inverse vibration problem

Consider for example, the free axial vibration of a rod, governed by the following differential equation

$$\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial w(x,t)}{\partial x}\right) = \rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2}. \tag{1.1}$$

Assuming harmonic motion, $w(x,t)=u(x)\sin(\omega t)$, (1.1) transforms to

$$\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) + \lambda\rho(x)A(x)u(x) = 0\,,\ \lambda=\omega^2. \tag{1.2}$$

With the inverse problem for the rod the aim is to determine the physical parameters $E(x)$, $A(x)$ and $\rho(x)$ from the knowledge of certain eigenvalues λ_i and eigenfunctions $u_i(x)$. The data needed for the reconstruction is supposed to be

generally available from experiments. In the absence of noise, the existence of a solution is thus guarantied. The main problem to be mastered here is the development of a procedure that may be used to numerically evaluate the unknown physical parameters.

In general, the solutions to direct problems lead to integration of the given data, which is a numerically stable process. The solution of the corresponding inverse problems however, generally involves differentiation. If the differentiation is carried out by means of numerical methods then the results may be highly sensitive to perturbations. Suppose that the Young's modulus of elasticity $E(x)$ is the only unknown in (1.2) and that all the other parameters are known constants. Then the modulus of elasticity $E(x)$ can be found from

$$E(x) = \frac{1}{\dfrac{\partial u}{\partial x}}\left[E(0)\frac{\partial u(0)}{\partial x} - \omega^2 \rho \int_0^x u(\xi) d\xi \right], \tag{1.3}$$

where $E(0)$ and $\dfrac{\partial u(0)}{\partial x}$ are determined by the boundary condition at $x=0$. It is evident from (1.3) that $E(x)$ depends on the derivative of $u(x)$ and this introduces the difficulty of obtaining a numerical solution, since numerical differentiation is sensitive to perturbation and noise. In practice, the data needed for the solution (natural frequencies and mode shapes) must be determined experimentally, inclusive of the measurement errors. Consequently, the resultant solutions and their attendant errors may lead to systems that cannot be physically realised.

Hence, when considering inverse problems it is important to establish a discrete model that closely matches the associated continuous model because this model presumably represents the actual dynamics of the physical systems.

To overcome the difficulty mentioned above and improve the consistency between the continuous model and its discrete representation, a higher order finite difference model is introduced in §3. A four-point approximation for the derivatives in the governing differential equations of motion is employed, which resulted in a pentadiagonal stiffness matrix. The inverse problem for an axially vibrating rod, modelled using the four-point finite difference approximation, is then solved using one eigenvalue, two eigenvectors and the total mass of the system. In §4 we discuss the inverse mode problem for a general discrete model of one-dimensional systems. This may be a finite difference, finite element, lumped-mass or any other discrete model. We will first develop a reconstruction procedure for the general case, and then apply the proposed solution for some typical models. In §5 multidimensional systems are considered and the corresponding inverse problem is solved for a simple model of two and three-dimensional vibrating structures. In order to experimentally verify the developed reconstruction algorithms, an experimental set-up consists of a model of a multi-storey building is caries out. It is shown that the results for the unknown physical parameters produced by numerical algorithms closely match the corresponding experimentally measured data.

A part of this work, described in detail in chapter 4, will be submitted in the *Journal of Sound and Vibration*, titled as

S. A. Burak and Y. M. Ram, "*Physical parameter reconstruction of vibratory models from eigendata*"

2 Background/Literature Review

2.1 Introduction

Inverse problems have become an important part in various theoretical and practical disciplines. In the following we present a basic background and a short survey of the main results that are found in the literature relevant to the subject. The discussion will be categorised according to the character of the systems, ie. continuous or discrete, and also according to the chosen sets of given data, ie. eigenvalues, mode shapes or mixed eigendata

2.2 Inverse problems for continuous systems

The research in this area was motivated by the analysis of the Sturm-Liouville equation. The fundamental results for the inverse problem of this equation are given by Gantmahker and Krein [23], Borg [12], Hald [41], [42] Gel'fand and Levitan [24] and others. Although essentially mathematical, these results may be associated with some standard problems in the vibration analysis. For example,

the *Sturm-Liouville* equation

$$y''(x)+[\lambda-q(x)]y(x)=0, \quad 0<x<l \tag{2.1}$$

subject to the boundary conditions

$$y'(0)-hy(0)=0, \quad y'(l)+Hy(l)=0, \tag{2.2}$$

analysed in Gantmakher and Krein [23] may be associated with the eigenvalue problem corresponding to the free vibration of a rod supported by springs at its ends. The axial vibration of such rod with constant density ρ and Young's modulus E is governed by the partial differential equation

$$\frac{\partial}{\partial x}\left(A(x)\frac{\partial u(x)}{\partial x}\right)+\lambda_1 A(x)u(x)=0,\ \lambda_1=\frac{\rho}{E}\omega^2, \tag{2.3}$$

which leads to the eigenproblem (2.1)-(2.2) by setting $q(x)=\dfrac{(\sqrt{A(x)})''}{\sqrt{A(x)}}$ and $y(x)=\sqrt{A(x)}u(x)$. Similarly, a special case of the Sturm-Liouville problem given by

$$u''(x)+\lambda\rho(x)u(x)=0 \tag{2.4}$$

may be used to describe the lateral vibrations of a string with varying density $\rho(x)$.

In the inverse problem for the Sturm-Liouville equation (2.1), the objective is to determine the potential $q(x)$ from the appropriate eigendata. Borg [12] has shown

that if h and H are given then $q(x)$ may be uniquely reconstructed from two sets of eigenvalues that correspond to two different boundary conditions. Hald [41] showed that only one spectral sequence is needed for the reconstruction in the case when $q(x)$ is symmetric about its mid point, ie. $q(x) = q(l - x)$. Hochstadt and Lieberman [46] extended this result and showed how to determine the potential $q(x)$ on one half of the interval $(0,l)$ if $q(x)$ is known on the other half. They proved that only one spectral sequence is needed for the reconstruction. In Gel'fand and Levitan [24], it is shown that not only the function $q(x)$ but also the constants h and H of the boundary conditions (2.2) may be uniquely determined from two sets of eigenvalues with the correct asymptotic form.

In Ram [69], the inverse problem for the continuous model of an axially vibrating rod, supported by two springs at its ends $x=0$ and $x=L$, was analysed. With the assumption of harmonic motion with frequency ω, the governing partial equation of motion transforms to the eigenvalue problem of the form

$$\begin{cases} \dfrac{d}{dx}\left(r(x)\dfrac{du(x)}{dx} \right) + \lambda\rho(x)u(x) = 0,\ \lambda = \omega^2,\ 0 < x < L \\ \alpha_0 u(0) - \beta_0 \dfrac{\partial u(0)}{\partial x} = 0 \\ \alpha_L u(L) - \beta_L \dfrac{\partial u(L)}{\partial x} = 0, \end{cases} \tag{2.5}$$

where $r(x)$ and $\rho(x)$ are the axial rigidity and the mass per unit length of the system. It has been shown that the unknown $r(x)$ and $\rho(x)$ may be reconstructed

uniquely from the knowledge of two eigenpairs and the total mass of the system. The necessary and sufficient conditions for the physical realizability of the system $(r(x), \rho(x) > 0)$ are given. In the case of a fixed-free configuration, the data needed for the solution consisted of one eigenvalue, two eigenfunctions and the total mass of the system.

Another important class of problems is associated with the fourth order differential equations. For example, the vibration of an Euler-Bernoulli beam may be described by the fourth order eigenvalue problem

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2u(x)}{dx^2}\right] - \lambda\rho A(x)u(x) = 0, \tag{2.6}$$

with appropriate boundary conditions. In the inverse problem for the beam, the aim is to determine the moment of inertia $I(x)$ and the cross-sectional area $A(x)$ from eigendata. Barcilon [2], [3] showed that $I(x)$ and $A(x)$ may be reconstructed from three sets of eigenvalues corresponding to the clamped-free, clamped-pinned and the clamped-sliding end conditions. The solution is based on the assumption that the given data are small perturbations of the eigenvalues corresponding to the uniform beam. McLaunghlin [54] improved this procedure by introducing the endpoint data into the analysis. In reference [29], Gladwell formulated the set of necessary and sufficient conditions on the spectral data to ensure a physically realistic solution for which the Euler-Bernoulli theory is valid. Gladwell, England and Wang [31] have extended this result and established further restrictions on the

input data for physical viability and validity of the Euler-Bernoulli theory. In [51] Loewe used a numerical procedure to evaluate the approximated coefficients of the Euler-Bernoulli beam with square cross sectional area, from the knowledge of (*i*) two spectral sets corresponding to the cantilever and clamped-supported end conditions, and (*ii*) the eigenvalues corresponding to the fixed-free configuration and the endpoint data of the eigenfunctions $\Phi_{2n-1}(1)$, $n=1, 2, ..., N$. The solution is based on a shooting method.

2.3 Discrete systems

In practice the systems under consideration are often complex structures and analytical methods cannot be effectively used, or when applied, they may lead to problems which cannot be solved in a closed form. In such cases, the practical technique is to introduce the finite differences or the finite elements, and obtain a numerical solution of the problem. In these methods, the real physical systems are analysed by regarding them as an assembly of basic elements with simple structure such as mass-spring systems, rods, beams or plates. The real continuous system is thus approximated by a discrete system with finite number of degrees of freedom. Mathematically, this means that the governing differential equation of motion is replaced by a system of n algebraic equations, and the problem is to determine the solution of these equations. One of the typical mechanical systems that may be naturally regarded as a discrete system is the mass-spring system oscillating in-line.

To demonstrate some of the concepts and basic procedures in inverse vibration

problems, we consider the two-degrees-of-freedom system shown in *Figure* 2-1. The system is undamped, fixed at one end and free to vibrate at the other. The free vibration of the system may be described by the following matrix equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{o}, \tag{2.7}$$

where

$$\mathbf{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \tag{2.8}$$

are the *stiffness* and *mass* matrices, respectively. Assuming harmonic motion, $\mathbf{x} = \mathbf{u}\sin\omega t$, equation (2.7) becomes

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{o}, \tag{2.9}$$

where $\lambda_i = \omega_i^2$, i = 1,2 are the eigenvalues of the system. In the inverse problem, the objective is to determine the physical properties k_1, k_2, m_1 and m_2. Since these

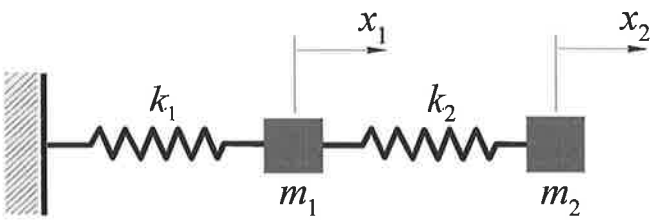

Figure 2-1 The mass-spring system in a fixed-free configuration

values are independent of each other (the only restriction is that they must be positive), we need to have four independent equations to evaluate them. There are a number of ways to provide these equations, but the preferred options are either (*i*) using mainly eigenvalues, or (*ii*) using mainly eigenvectors. We call the problems associated with (*i*) *inverse eigenvalue problems* and the problems associated with *(ii) inverse mode problems*.

(*i*) Inverse eigenvalue problem

We continue with our example of *Figure* 2-1. The system (2.9) is a homogeneous system of two algebraic equations. In order to have a non-trivial solution, $\mathbf{u}\neq\mathbf{0}$, to this system, the coefficient matrix must be singular. This yields the characteristic equation

$$\lambda^2 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]\lambda + \frac{k_1 k_2}{m_1 m_2} = 0\,. \tag{2.10}$$

The two roots of (2.10) may be determined from

$$\lambda_1 + \lambda_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \tag{2.11}$$

and

$$\lambda_1 \lambda_2 = \frac{k_1 k_2}{m_1 m_2}\,. \tag{2.12}$$

If the values for both λ_1 and λ_2 are known then equations (2.11) and (2.12) may be used to determine the stiffnesses and the masses. However, they are not sufficient for the determination of four unknown parameters k_1, k_2, m_1 and m_2. To provide an addition equation the configuration of the system may be changed in such a way that the mass m_2 on right end is fixed, as shown in *Figure* 2-2. This will force the system to vibrate in a different way, characterised by the eigenvalue

$$\lambda_3 = \frac{k_1 + k_2}{m_1}. \tag{2.13}$$

The fourth equation needed for the reconstruction may be provided from an additional requirement that the total mass, or the total stiffness, is known. If we assume the knowledge of the total mass of the system M, then

$$M = m_1 + m_2 . \tag{2.14}$$

The four equations (2.11)-(2.14) are sufficient for the determination of k_1, k_2, m_1 and m_2 uniquely. However, since the physical parameters k_1, k_2, m_1 and m_2 must

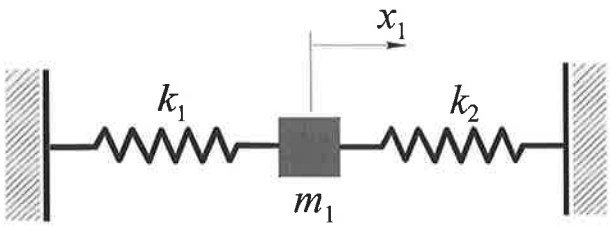

Figure 2-2 The mass-spring system in a fixed-fixed boundary conditions

be positive, we still need to determine conditions on the data which guarantee positive solutions. The following analysis will help us to specify these conditions.

From equation (2.11), (2.12) and (2.13) we have

$$\frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \frac{k_1 + k_2}{m_1}, \tag{2.15a}$$

$$\frac{k_1}{m_1} = \frac{m_2}{k_2}\lambda_1\lambda_2 \tag{2.15b}$$

and

$$\frac{k_2}{m_2} = \lambda_3 - \frac{k_1}{m_1}. \tag{2.15c}$$

Using (2.13) the above relations may be rearranged as

$$\frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \lambda_3, \tag{2.16}$$

$$\frac{k_1}{m_1} = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} \tag{2.17}$$

and

$$\frac{k_2}{m_1} = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 - \lambda_3}. \tag{2.18}$$

Since k_1, k_2, m_1 and m_2 must be positive, it follows from (2-16)-(2.18) that the

eigenvalues λ_1, λ_2 and λ_3 must be such that the following is satisfied

$$0 < \lambda_1 < \lambda_3 < \lambda_2 . \tag{2.19}$$

Condition (2.19) is called *the interlacing property for eigenvalues*, and it represents the necessary condition for the physical realizability of the system. In the case of a mass-spring system with many degrees of freedom the above methodology may be generalised, as done in Gladwell [28].

(*ii*) Inverse mode problem

Suppose that one eigenvalue λ_1, two eigenvectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ the total mass M of the system are given, where the eigenvalue λ_1 is associated with $\mathbf{u}^{(1)}$. The eigenvalue problem (2.8) must be satisfied for both eigenvectors, ie.

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{u}^{(1)} = \mathbf{o}, \tag{2.19}$$

$$(\mathbf{K} - \lambda_2\mathbf{M})\mathbf{u}^{(2)} = \mathbf{o}, \tag{2.20}$$

where λ_2 is the eigenvalue corresponding to $\mathbf{u}^{(2)}$. The last rows of (2.19) and (2.20) give the following two equations

$$-k_2u_1^{(1)} + k_2u_2^{(1)} - \lambda_1 m_2 u_2^{(1)} = 0, \tag{2.21a}$$

$$-k_2u_1^{(2)} + k_2u_2^{(2)} - \lambda_2 m_2 u_2^{(2)} = 0, \tag{2.21b}$$

or, written in the matrix form

$$\begin{bmatrix} (u_2^{(1)} - u_1^{(1)}) & -\lambda_1 u_2^{(1)} \\ (u_2^{(2)} - u_1^{(2)} & -\lambda_2 u_2^{(2)} \end{bmatrix} \begin{Bmatrix} k_2 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{2.22}$$

System (2.22) is homogenous and may have a non-trivial solution only if the determinant of the coefficient matrix is equal to zero. Evaluating this determinant, we obtain

$$\lambda_2 = \lambda_1 \frac{u_2^{(1)}(u_2^{(2)} - u_1^{(2)})}{u_2^{(2)}(u_2^{(1)} - u_1^{(1)})}. \tag{2.23}$$

which determines the second eigenvalue of the system. Providing that this conditions is satisfied, the system (2.22) may be used to determine the ratio k_2/m_2 from

$$\frac{k_2}{m_2} = \lambda_1 \frac{u_2^{(1)}}{u_2^{(1)} - u_1^{(1)}}. \tag{2.24}$$

We now use the first two rows of (2.19) and (2.20) and evaluate the ratios k_1/m_2 and m_1/m_2 by

$$\begin{Bmatrix} k_1 / m_2 \\ m_1 / m_2 \end{Bmatrix} = \begin{bmatrix} u_1^{(1)} & -\lambda_1 u_1^{(1)} \\ u_1^{(2)} & -\lambda_2 u_1^{(2)} \end{bmatrix}^{(-1)} \begin{Bmatrix} (u_2^{(1)} - u_1^{(1)}) \\ (u_2^{(2)} - u_1^{(2)}) \end{Bmatrix} \frac{k_2}{m_2}, \tag{2.25}$$

because k_2/m_2 and λ_2 are already determined by (2.24) (2.23). At the end of the procedure we take into account the knowledge of the total mass M and reconstruct the actual physical parameters using

$$k_1 = \left(\frac{k_1}{m_2}\right)\frac{M}{\frac{m_1}{m_2}+1},\quad k_2 = \left(\frac{k_2}{m_2}\right)\frac{M}{\frac{m_1}{m_2}+1},\quad m_1 = \left(\frac{m_1}{m_2}\right)\frac{M}{\frac{m_1}{m_2}+1},\quad m_2 = M - m_1 \quad (2.26)$$

Expressions (2.26) are the unique solution of the posed inverse problem. The problem of deriving the necessary and sufficient conditions on the given data to ensure the positive physical parameters, however, is now more complex. It may be shown that in this simple case these conditions are

- λ_1 and M must be positive and real
- if $u_2^{(1)} > 0$ and $u_2^{(2)} > 0$, then $u_1^{(1)} > u_2^{(1)}$ and $u_1^{(2)} < 0$,
- if $u_2^{(1)} > 0$ and $u_2^{(2)} < 0$, then $u_1^{(1)} < u_2^{(1)}$ and $u_1^{(2)} > 0$.
- if $u_2^{(1)} < 0$ and $u_2^{(2)} > 0$, then $u_1^{(1)} > u_2^{(1)}$ and $u_1^{(2)} < 0$.
- if $u_2^{(1)} < 0$ and $u_2^{(2)} < 0$, then $u_1^{(1)} > u_2^{(1)}$ and $u_1^{(2)} > 0$.

In Gladwell [27] these conditions are analysed for the general mass-spring systems of any order n. It is shown that the necessary conditions for $\mathbf{u}^{(i)}$ to be the ith mode of a n order fixed-free mass-spring system are given by

- $S_u^+ = S_u^- = S_w^+ = S_w^- = i - 1$, and
- $u_1 w_1 > 0$,

where S_u^+ ans S_u^- represent the greatest and the least values of the number of sign

interchanges of $\mathbf{u}^{(i)}$, respectively, and where $w_n = u_n - u_{n-1}$. The necessary and sufficient conditions that *two* vectors **u** and **v** be the ith and the jth modes of a mass-spring system, $i<j$, are

- $S_u = S_w < S_v = S_z$,
- $v_N w_N > 0$,
- $s_{N-1} > 0$,
- for each value of n, p_n, , q_n and r_n have the same sign and this sign need not be the same for all $n \in [1, N-1]$,

where

$$s_{N-1} = u_{N-1} v_N - u_N v_{N-1}, \tag{2.27a}$$

$$p_n = u_n v_N w_N z_n - u_N v_n w_n z_N, \tag{2.27b}$$

$$q_n = u_n v_N w_N z_{n+1} - u_N v_n w_{n+1} z_N \tag{2.27c}$$

and

$$r_n = w_n z_{n+1} - w_{n+1} z_n. \tag{2.27d}$$

In the case that **u** and **v** are the neighbouring eigenvectors corresponding to the ith and the $(i+1)$th modes of the mass-spring system, the conditions are simplified to

- $v_N w_N > 0$,
- $s_{N-1} > 0$,
- $p_k > 0,\ q_k > 0,\ r_k > 0,\ k = 1,2,...,N$.

2.3.1 Inverse eigenvalue problems for discrete systems

The problem of reconstructing the matrices **K** and **M** from equation (2.9) is equivalent to the problem of finding a *Jacobi* matrix **A** satisfying

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{o}, \tag{2.28}$$

where $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ and $\mathbf{x} = \mathbf{M}^{1/2}\mathbf{u}$. This problem was posed by Gantmakher and Krein [23] in their study of the transverse vibration of a set of masses on a string.

Hochstadt [45] analysed the similar problem: *find a Jacobi matrix A of the form*

$$\mathbf{A} = \begin{bmatrix} a_1 & -b_1 & & & \\ -b_1 & a_2 & -b_2 & & \\ & . & . & . & \\ & & -b_{n-2} & a_{n-1} & -b_{n-1} \\ & & & -b_{n-1} & a_n \end{bmatrix}, \quad b_i > 0,\ i = 1,2,...,n \tag{2.29}$$

satisfying $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\mathbf{o}$, *such that A has the prescribed set of eigenvalues* $\lambda_1, \lambda_2, ...,\lambda_n$. He has shown that it is possible to reconstruct at most one matrix **A** that have the prescribed set of eigenvalues if the eigenvalues $\mu_1, \mu_2, ...,\mu_{n-1}$ of the matrix $\mathbf{A}^*$

obtained by deleting the last row and column of **A** are also given. It was shown that the necessary condition for such reconstruction is the interlace relation between the two spectral sequences

$$0<\lambda_1<\mu_1<\lambda_2<\ldots<\mu_{n-1}<\lambda_n. \tag{2.30}$$

Similar inverse problem was studied by Hald in his analysis of the discrete Sturm-Liouville equation. In [42] he developed the basic theory of existence and uniqueness of the solution for the inverse problem of Jacobi matrices. He also gave a solution to the problem of reconstructing an n-th order Jacobi matrix from the two sets of eigenvalues, but it has been shown that the proposed solution was numerically unstable. The reconstruction procedure consisted of constructing the principal minors of $\mathbf{A}$-$\lambda\mathbf{I}$

$$P_0=1, P_1(\lambda)=a_1-\lambda, P_2(\lambda)=(a_1-\lambda)(a_1-\lambda)-b_1^2,\ldots P_2(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I}), \tag{2.31}$$

which satisfy the relation

$$P_{i+1}(\lambda)=(a_{i+1}-\lambda)P_i(\lambda)-b_i^2P_{i-1}(\lambda),\ i=1,2,\ldots,n-1, \tag{2.32}$$

and forming the polynomials

$$p_n(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i) \text{ and } p_{n-1}(\lambda)=\prod_{i=1}^{n-1}(\lambda-\mu_i). \tag{2.33}$$

It is shown that these two polynomials are the nth and (n-1)th principal minors of **A**, so they satisfy the recurrence relation (2.32). The next step is to form $p_{n-2}(\lambda)$

from (2.33) and evaluate the parameters a_n and b_{n-1}. Then, using $p_{n-1}(\lambda)$ and $p_{n-2}(\lambda)$ he form the polynomial $p_{n-3}(\lambda)$ and evaluate a_{n-1} and b_{n-2}. Continuing in the same way for other rows of this procedure produced all a_i and b_i which may be used to reconstruct the matrix **A** using (2.29) . This generalise the results of our simple example in section 2.3. De Boor and Golub [11] found that the above procedure could fails to be unstable and developed an alternative direct algorithm, which is numerically stable.

Another method for reconstructing a Jacobi matrix was given by Boley and Golub [10]. Let the matrix **A** be partitioned in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & v_1^T \\ v_1 & \overline{\mathbf{A}} \end{bmatrix}, \tag{2.34}$$

where $\overline{\mathbf{A}}$ is a (n-1) by (n-1) leading principal submatrix of **A**. It is shown that if the eigenvalues $\{\lambda\}_1^n$ and $\{\mu\}_1^{n-1}$ of **A** and $\overline{\mathbf{A}}$ are known, then the first row (q_{11} ...q_{1n}) of the normalised eigenvector matrix **Q** of **A** may be determined by these sets applying the formula

$$q_{1i}^2 = \frac{\prod_{j=1}^{n-1}(\mu_j - \lambda_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(\lambda_j - \lambda_i)}. \tag{2.35}$$

Using these quantities and the *Lanczos* algorithm they reconstructed the matrix **A** in a stable numerical procedure.

Ram and Caldwell [68] analysed a multiple connected mass-spring system where each mass could be connected to every other, as shown for example in *Figure* 2-3. The data needed for the reconstruction of such systems may be obtained from the knowledge of its n eigenvalues, (n-1) eigenvalues corresponding to the same system but with a fixed mass m_n, (n-2) eigenvalues of the system with fixed masses m_n and m_{n-1}, and so on down to the single eigenvalue corresponding to the system with all but m_1 fixed. They provided a way to reconstruct the stiffness and mass matrices from this set of eigenvalues and the total mass of the system. In the reconstruction procedure they first transformed the eigenvalue problem associated with the system $(\mathbf{K}-\lambda\mathbf{M})\mathbf{v}=0$ to the standard form $(\mathbf{E}-\lambda\mathbf{I})\mathbf{u}=0$ by setting $\mathbf{E}=\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ and $\mathbf{v}=\mathbf{M}^{-1/2}\mathbf{u}$. Next, they showed how to reconstruct all possible submatrices of $\mathbf{E}$ with given eigenvalues and finally they developed an reconstruction algorithm to evaluate $\mathbf{K}$ and $\mathbf{M}$ from $\mathbf{E}$. The necessary condition on the given data to ensure a positive solution is that successive sets of eigenvalues interlace.

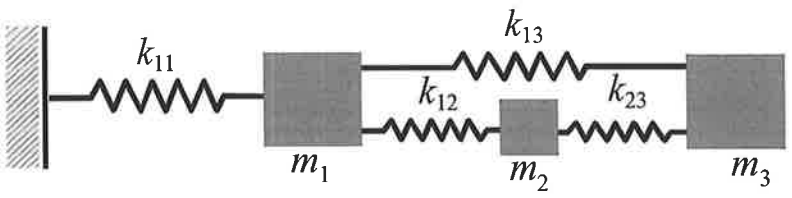

Figure 2-3 A multiple connected mass-spring system

Analysing similar multiple connected mass-spring systems, Gladwell and Movahedy [36] provided a set of necessary and sufficient conditions for the case where $n=3$. In the case of the fully connected three degrees of freedom system they have shown that it is possible to reconstruct:

a) two different systems if $\frac{1}{R} < \frac{\mu_2 - \mu_1}{\lambda_1 - \nu_2} < R$, $R > 1$

b) one system if $\frac{\mu_2 - \lambda_1}{\lambda_1 - \nu_1} < \frac{1}{R}$ or $\frac{\mu_2 - \lambda_1}{\lambda_1 - \nu_1} > R$

c) one system if $\frac{\mu_2 - \lambda_1}{\lambda_1 - \nu_1} < R$ or $\frac{\mu_2 - \lambda_1}{\lambda_1 - \nu_1} > R$, $R \le 1$, and

d) no system if $R < \frac{\mu_2 - \lambda_1}{\lambda_1 - \nu_1} < \frac{1}{R}$, $R \le 1$,

where R is defined by

$$R = \frac{(\mu_1 - \nu_1)(\nu_2 - \mu_1)(\nu_3 - \mu_1)}{(\mu_2 - \nu_1)(\mu_2 - \nu_2)(\nu_3 - \mu_2)}, \quad \mu_i, \nu_i \text{ - given eigenvalues.} \tag{2.36}$$

It is shown that for the case with $n \ge 4$ the problem becomes too complex and some conditions are given for $n=4$.

Friedland, Nocedal and Overton [22] formulated and analysed an essentially mathematical inverse eigenvalue problem for a discrete symmetric matrix $\mathbf{A}$ represented in the form $\mathbf{A}(c) = \mathbf{A}_0 + \sum_{i=1}^{n} c_i \mathbf{A}_i$. The problem is to find c_i from the given set of the real numbers $\lambda_1^* \le \lambda_2^* \le \ldots \lambda_n^*$ such that $\lambda_i(c) = \lambda_i^*$ where λ_i, $i=1,2,$

...,n are the eigenvalue of $\mathbf{A}(c)$. A similar problem was analysed by Downing and Householder [19] for the case where the eigenvalues are strictly distinct $\lambda_1(c) < \lambda_2(c) < ... < \lambda_n(c)$. They provided a numerical algorithm for solving the problem by applying the Newton's method.

Friedland *et al.* [22] have extended the domain of the problem by taking into account the case where some eigenvalues may be multiple, for example $\lambda_1^* = \lambda_2^* = ... = \lambda_k^* < \lambda_{k+1}^* < ... < \lambda_n^*$. They modified the proposed method described by Downing and Householder by using an alternative method of computing the eigenvectors in each step. The rate of convergence of the modified problem is quadratic.

Chu [14] discusses a solution to the additive inverse eigenvalue problems, which is a special case of the problem analysed by Friedland, Nocedal and Overton: *Given real numbers* $\lambda_1^* \le \lambda_2^* \le ... \lambda_n^*$, *find c such that* $\lambda_i(c) < \lambda_i^*$, $i = 1,2, ..., n$ *of a symmetric matrix of the form* $A_i = e_i e_i^T$ *where* e^i *is the i-th unit vector.* The suggested solution to the problem was based on the homotopy method.

Barcilon [2] analyses the inverse problem for a discrete non-homogenuous beam. Applying a finite difference approximation on the fourth order differential equation of motion for a vibrating beam

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \qquad (2.37)$$

leads to a pentadiagonal symmetric matrix. In this paper, the author generalised the previous result of Hochtadt for the inverse problem of a tridiagonal matrix and demonstrates that the solution, if it exists, may be found from the given sets of three spectra: one corresponding to the clamped-clamped, clamped-supported and clamped-free end conditions.

The inverse problem for a discrete model of an Euler-Bernoulli vibrating beam is also analysed by Gladwell [25]. In this work, the continuous model of the beam is approximated by the model consisting of a set of rigid rods connected to each other by springs of torsional stiffnesses k_i (see *Figure* 2-4). Introducing transformations $\mathbf{v}=\mathbf{M}^{1/2}\mathbf{u}$ and $\mathbf{A}=\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}$, the eigenvalue problem for the beam transforms to the standard form $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, where $\mathbf{A}$ is a sign oscillatory matrix. Keeping the left end of the system fixed while applying different boundary conditions on the right, results in different modes of vibration, with several sets of natural frequencies. Analysing the dynamic behaviour of such a system, the author confirmed the previously known fact that the corresponding

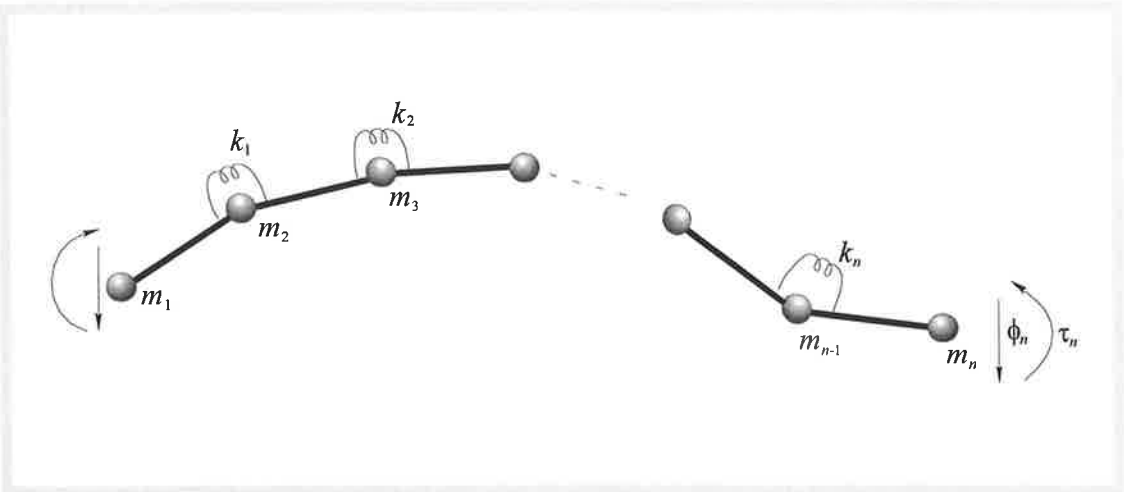

Figure 2-4 The Discrete model of a vibrating beam

sets of eigenvalues must interlace for the physically realisable systems. The additional conditions on the data have been found by requiring that all submatrices of the coefficient matrix must be non-singular. However, it has been shown that realistic results may be expected only if the given data slightly differs from the data corresponding to a uniform case.

2.3.2 Inverse mode problems for discrete systems

One of the alternative ways to tackle the problem of determining the physical parameters is to take into an account the mode shapes (eigenvectors) of the system. These quantities may be easily and accurately obtained using the experimental modal analysis technique, and represent valuable data about the dynamics of the systems.

Gladwell [28] has shown that the lumped-mass systems with tridiagonal stiffness and diagonal mass matrices may be reconstructed from the knowledge of two vibrating modes and their corresponding eigenvalues. The necessary and sufficient conditions on the input data that ensure the existence of such a system are provided.

Ram and Gladwell [71] have analysed discrete systems where both, the stiffness and the mass matrices are tridiagonal. A finite element model for the axially vibrating rod is one of such systems. It is shown that the mass and stiffness parameters of the model may be reconstructed using one eigenvalue, two

eigenvectors and the total mass. Since the corresponding eigenvalue equation must be satisfied for all eigenpairs, the use of two eigenvectors resulted in two matrix equations. One of the eigenvalues is given while the other is determined from the last two rows of these equations. This also produced the ration k_n/m_n, where k_n and m_n are the stiffness and mass of the nth finite element. Then, using other n-1, n-2, ...,1 rows they determined the ratios k_i/m_n and m_i/m_n for i= n-1, n-2, ...,1. The final step was to use the given total mass and to evaluate actual k_i and m_i from these ratios.

Ram [70] used a similar approach in analysing in the inverse mode problem for a discrete model of a vibrating beam. Again, the solution for the unknown physical parameters of the system was based on the knowledge of two eigenvectors, one eigenvalue and the total mass of the system. If the position of measurements is unknown, then three eigenvectors, one eigenvalue and the total mass are needed for the reconstruction of the model and its measurement points. The physical model used is a mass-spring-rod, ie. a finite-difference model. In this work, the author specified the conditions needed for the unique solution, but the problem of obtaining sufficient conditions on the data was not addressed. The question of the impact of noise and perturbation was also studied. The sensitivity of the solution to such perturbations was reduced by the introduction of the additional eigendata and as a result the model was reconstructed as an optimal solution using the least squares methodology. The presented numerical examples clearly demonstrate the effectiveness of such an approach. The problem is however that it is not

practically possible to obtain so much data with the required accuracy, particularly if it is known that the difference between discrete models and real (continuous) systems is significant for the higher modes of vibrations.

Ahmedian, Gladwell and Ismail [1] analysed finite element models of some simply vibrating systems. An effective regularisation method was used to obtain a particular solution for the unknown model parameters from a class of possible solutions, in the sense that the solution is the nearest to *a priori* set of values for these parameters. Their analysis started with the finite element model of a general vibrating system, but then they limited the analysis to simpler special systems such as rods or beams. In addition, the finite element model of a cantilever beam was studied using the quadratic B-spline interpolations. As a result, the stiffness and mass matrices were both pentadiagonal symmetric and this increased the sensitivity to perturbation on the input data. Regularisation was then used and the inverse problem was successfully solved. Given numerical experiments have proved this. However, many questions were left open in this work. Problems such as the possibility of generalising the procedure on other, or at least similar mechanical systems needed to be answered Also, the open question is will the reconstruction be successful if the model has many degrees of freedom.

The discrete model of a cantilever beam in flexural vibration was analysed by Gladwell, Willms, He and Wang [32]. They have demonstrated that the discrete model of such a system may be reconstructed from one mode and they also provided a complete set of necessary and sufficient conditions on that mode to

ensure that the physical parameters are always positive. However, the solution was not unique and the question is if the rotational displacement may be neglected in a real situation.

2.4 Two and three dimensional problems

Inverse problems associated with two and three-dimensional models of vibrating systems have been studied mainly in recent times and very few papers are currently published in this area. The reason for this is clearly evident: the complexity of such systems is much higher and both direct and inverse problems associated with them are more difficult.

In a recent paper given by Gladwell and Zhu [35], it was considered an undamped two-dimensional system consisting of n masses and connected by springs only between neighbours. Although relatively simple, this assumption still holds for many physical systems in the engineering practice. For example, in a high floor building structure, the floors are connected in such a way. The resulting stiffness matrix is of block tridiagonal form, while the mass matrix is block diagonal. Each block in these matrices is of order $n \times n$, so both K and M are $n^2 \times n^2$ square matrices. In order to reconstruct their coefficients, the authors applied the Forsyth algorithm assuming that the appropriate spectral sets are known.

Similar mathematical problems are analysed by Barcilon [5].

3 An inverse mode problem for a higher order finite difference model of a rod

3.1 Introduction

As mentioned in the previous chapter, the analytical solutions for continuous models in vibration may be obtained only for some simple systems, such as rods, beams and membranes of simple geometry. If a closed form solution cannot be found, then the standard approach is to discretize the physical system with an appropriate model and obtain an approximated solution. The most widely used techniques applied for such purposes are the methods of finite differences and finite elements.

In this chapter the method of finite differences is applied as an analytical model of an axially vibrating rod. It was demonstrated in Ram and Gladwell [71] that the inverse problem for the rod might be solved by using the finite elements or, alternatively, the standard two-point finite difference approximation. The accuracy of these models is often acceptable, and it has been shown on many examples that they may give relatively accurate solutions for the natural frequencies and mode shapes, particularly for the lower modes of vibrations. In the inverse approach however, the accuracy of these models may be insufficient

and may produce poor or physically non-acceptable results. In practice the data needed for the solution of an inverse problem, ie. the natural frequencies and mode shapes, must be determined experimentally and therefore may introduce measurement errors. In *Figure* 3-1 a typical situation resulting from the determination of the natural frequencies for an unspecified vibratory system is shown. In this case, the natural frequencies corresponding to the continuous model are close to the experimental observations, which means that the conditions that are stated for the validity of the model are satisfied. The natural frequencies corresponding to a discrete model of the system however, are not so accurate, particularly for the higher modes of vibration.

In the inverse approach the actual analytical model must be known in advance. If the identified model corresponds to an inaccurate discrete one, then the

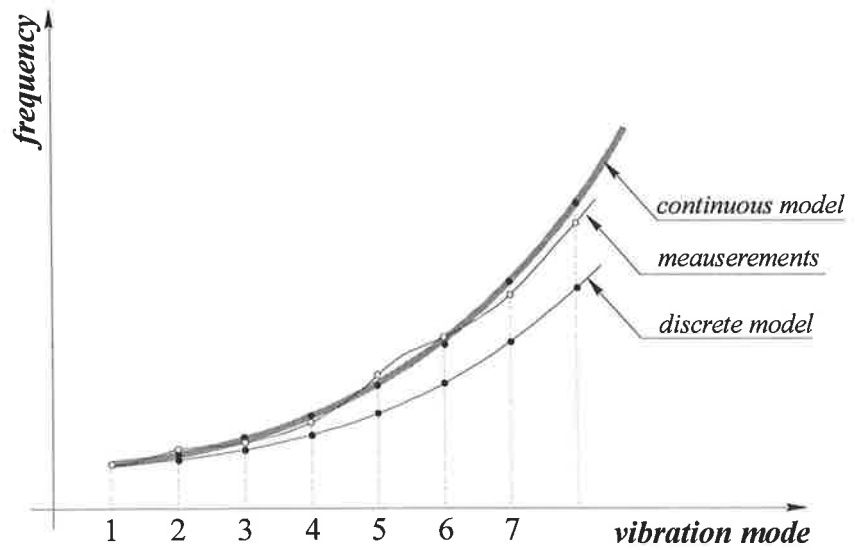

Figure 3-1 The discrepancy between the theoretical models and measurements

reconstruction procedure may fail or produce significant errors in the solution. It is therefore clear that in the inverse problems the consistency and accuracy of the analytical models are of great importance. In the case of finite difference models, the accuracy of the approximation may be increased if a higher order finite difference scheme, based on four or more points, is used. For example, the discretisation errors of a four-point approximation for the first and second derivatives are of order $O(h^4)$, h being the length of the segment, whereas in the two-point scheme the errors are of order $O(h^2)$.

In the following we will apply a higher order finite difference scheme on the differential equation of a longitudinally vibrating rod. Latter in the chapter, we will state the inverse problem for this model and develop a reconstruction procedure for the unknown physical parameters of the system.

3.2 Standard models for vibrating systems

Generally, each physical system can be modelled by using some of its physical properties and applying the adequate physical laws that bind them mutually. For vibratory systems, these properties are usually the mass, stiffness and damping. The resulting equations of motion are then either partial differential equations for the continuous systems, or sets of ordinary differential equations for discrete models.

For example, the longitudinal vibration of a thin elastic rod shown in *Figure* 3-2, may be described by the following differential equation

$$\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial w(x,t)}{\partial x}\right) = \rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2}, \tag{3.1}$$

where

$w(x,t)$ is the axial displacement of an element dx,

$E(x)$ is the Young's modulus of elasticity,

$A(x)$ is the cross sectional area, and

$\rho(x)$ is the mass per unit length.

Similarly, the torsional vibration of a shaft shown in *Figure* 3-3, is governed by the equation

$$\frac{\partial}{\partial x}\left(I_p G\frac{\partial \vartheta(x,t)}{\partial x}\right) = \rho I_p \frac{\partial^2 \vartheta(x,t)}{\partial t^2}, \tag{3.2}$$

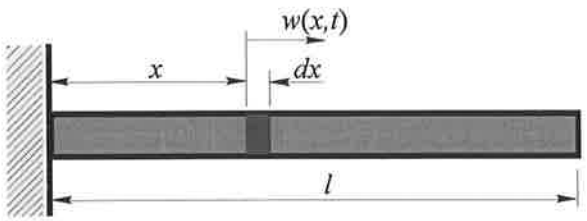

Figure 3-2 The longitudinally vibrating rod

where:

I_pG is the torsional stiffness of the shaft,

I_p is the polar moment of inertia of the cross section, and

G is the shear modulus of the material.

For complete description of the motion, the adequate boundary and initial conditions need to be specified. Although the precise information about these conditions is usually difficult to determine in practice, we can still derive them with high degree of accuracy in typical situations. For instance, in the case of a vibrating rod, the fixed-free boundary conditions are given by

$$w(x,t)\big|_{x=0} = 0 \text{ (no displacements at the fixed end)}, \tag{3.3}$$

and

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=l} = 0 \text{ (no stress at the free end)}, \tag{3.4}$$

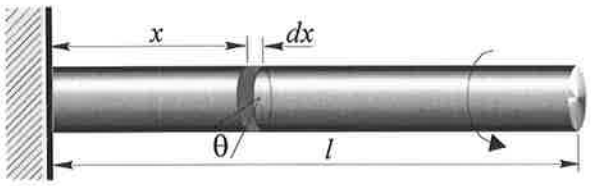

Figure 3-3 The torsional vibration of a shaft

where l is the length of the rod.

The initial conditions for the rod may be generally expressed as

$$w(x,t)\Big|_{t=0} = f(x) \text{ and } \frac{\partial w(x,t)}{\partial t}\Big|_{t=0} = g(x), \tag{3.5}$$

which are the mathematical expressions for the initial displacement and initial velocity. The assumption of the harmonic motion about the neutral axes of the system,

$$w(x,t) = u(x)\sin(\omega t), \tag{3.6}$$

transforms equation (3.1) to the eigenvalue problem

$$\begin{cases} \dfrac{d}{dx}\left(E(x)A(x)\dfrac{du(x)}{dx} \right) + \lambda\rho(x)A(x)u(x) = 0 \\ \\ u(x)\Big|_{x=0} = 0 \text{ and } \dfrac{du(x)}{dx}\Big|_{x=L} = 0 \end{cases}, \tag{3.7}$$

where $\lambda = \omega^2$ is a constant parameter, called an eigenvalue.

The solution of the above system requires the knowledge of the physical properties of the rod $A(x)$, $\rho(x)$ and $E(x)$ and yields an infinite set of eigenvalues λ_i and their corresponding eigenfunctions $u_i(x)$. In the case of a uniform rod, the physical parameters are constant and the problem may be solved in a closed form. If $A = A(x)$, $\rho = \rho(x)$ or $E = E(x)$ however, it is mathematically difficult, or sometimes

impossible, to solve the problem analytically. In such cases the practical technique is to apply some of the approximate methods such are the method of finite differences or finite elements.

In the method of finite differences, the derivatives that appear in the differential equations are approximated using the adequate finite difference schemes. In the standard two-point model, the first and second derivatives of a continuous function $f(x)$ are approximated by

$$\frac{df(x)}{dx} = \frac{f(x+h)-f(x-h)}{2h} + O(h^2) \tag{3.8a}$$

and

$$\frac{d^2 f(x)}{dx^2} = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + O(h^2), \tag{3.8b}$$

where h is the length of the finite difference segment. In order to apply this approximation on the eigenvalue problem (3.7), we imagine that the system is divided into n equal subintervals, as it is shown in *Figure* 3-5. Let $x_0=0$, $x_i = x_0+ih$

Figure 3-5 The finite difference model of a vibrating rod

for $i=1, 2, ..., n\text{-}1$ and $x_n=L$, where L is the length of the rod. Then, using (3.8) we obtain

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) \cong \frac{E_{i+1}A_{i+1}(u_{i+1}-u_i)-E_iA_i(u_i-u_{i-1})}{h^2}. \tag{3.9}$$

At the end points $i=0$ and $i=n$, the boundary conditions (3.7) must be satisfied. In the finite difference formulation, they become

$$u_0 = 0, \quad \frac{u_{n+1}-u_n}{h} = 0. \tag{3.10}$$

Introducing (3.9) and (3.10) in the eigenvalue problem (3.7) results in the following set of algebraic equations

$$-k_iu_{i-1}+(k_i+k_{i+1})u_i-k_{i+1}u_{i+1}-\lambda m_iu_i = 0, \tag{3.11}$$

where

$$k_i = \frac{E_iA_i}{h}, \; m_i = h\rho_iA_i, \; i = 1,2, ..., n \tag{3.12}$$

are the discrete representations of the stiffness and mass parameters of the i-th element. Introducing the eigenvector $\mathbf{u}=\{u_1\ u_2\ ...\ u_n\}^T$ and the stiffness and mass matrices by

$$\mathbf{K} = \begin{bmatrix} (k_1+k_2) & (-k_2) & & & \\ (-k_2) & (k_2+k_3) & (-k_3) & & \\ & \ddots & & & \\ & & & (k_{n-1}+k_n) & (-k_n) \\ & & & (-k_n) & (k_n) \end{bmatrix} \tag{3.13}$$

and

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{n-1} & \\ & & & & m_n \end{bmatrix}, \tag{3.14}$$

the above system may be written in the matrix form as

$$\mathbf{Ku} = \lambda \mathbf{Mu}. \tag{3.15}$$

The stiffness matrix **K** in this model is tridiagonal, symmetric and positive definite, while the mass matrix **M** is diagonal and positive. We note (3.15) and (3.14) have an identical form as the stiffness and mass matrices of a mass-spring system, discussed in the previous chapter. As such, they are convenient in computations and standard numerical algorithms may be applied. It has been shown in Ram and Gladwell [71] that the stiffness and mass matrices of the *finite element* model of the rod may be reconstructed from two eigenvectors and one eigenvalue. In the following example we demonstrate this solution in the case where the rod is modelled using the above *finite difference* approximation.

3.2.1 Example 3-1

For $n=5$, suppose that the following data is given

$$\lambda_1 = 10, M = 20,$$

$$\mathbf{u}^{(1)} = \{1 \quad 3 \quad 5 \quad 6 \quad 6.5\},$$

$$\mathbf{u}^{(2)} = \{3 \quad 5 \quad 1 \quad -3 \quad -7\},$$

where λ_1 is an eigenvalue, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ are two eigenvectors and M is the total mass of the system. Assuming that the system is represented by (3.15), the problem is to reconstruct the physical parameters k_i and m_i, $i=1,2, ..n$, and via (3.14) and (3.15), the stiffness and mass matrices. The eigenvalue problem (3.15) must be satisfied for both eigenvectors. This gives

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{u}^{(1)} = \mathbf{o} \tag{3.16a}$$

and

$$(\mathbf{K} - \lambda_2\mathbf{M})\mathbf{u}^{(2)} = \mathbf{o}, \tag{3.16b}$$

where λ_2 is the eigenvalue corresponding to $\mathbf{u}^{(2)}$. The first step in the reconstruction procedure is to eliminate this eigenvalue from the system. In order to do this we start, as in [71], from the last rows of (3.16a) and (3.16b). Written in the matrix form, these two equations yield

$$\begin{bmatrix} (u_5^{(1)} - u_4^{(1)}) & -\lambda_1 u_5^{(1)} \\ (u_5^{(2)} - u_4^{(2)} & -\lambda_2 u_5^{(2)} \end{bmatrix} \begin{Bmatrix} k_5 \\ m_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.17}$$

The physical parameters k_5 and m_5 must be positive, so the determinant of the coefficient matrix in (3.17) must vanish. This gives

$$\lambda_2 = \lambda_1 \frac{u_5^{(1)}(u_5^{(2)} - u_4^{(2)})}{u_5^{(2)}(u_5^{(1)} - u_4^{(1)})} = 89.4118. \tag{3.18a}$$

Finding the eigenvalue λ_2 the system (3.17) may be used to determine the ratio k_5/m_5, ie.

$$\frac{k_5}{m_5} = \lambda_1 \frac{u_5^{(1)}}{u_5^{(1)} - u_4^{(1)}} = 380. \tag{3.18b}$$

The next step is to consider the fourth rows of (3.16a) and (3.16b). Dividing (3.16a) and (3.16b) by m_5 we use these two equations to find the ratios $\frac{k_4}{m_5} = 744.6465$ and $\frac{m_4}{m_5} = 3.2626$. Then, setting $i=3,2$ and 1, we solve the third, the second and the first rows of (3.16a) and (3.16b) and evaluate: $\frac{k_3}{m_5} = 481.5974$, $\frac{m_3}{m_5} = 1.6811$; $\frac{k_2}{m_5} = 574.1823$, $\frac{m_2}{m_5} = 2.6453$ and $\frac{k_1}{m_5} = 1244.8$, $\frac{m_1}{m_5} = 4.8203$. At the end of the procedure, we use the knowledge of the total mass M and determine the actual physical parameters by

$$k_1 = \left(\frac{k_1}{m_5}\right) \frac{M}{\sum_{i=1}^{5} \frac{m_i}{m_5}} = 92.8286, \quad k_2 = \left(\frac{k_2}{m_5}\right) \frac{M}{\sum_{i=1}^{5} \frac{m_i}{m_5}} = 42.8196,$$

$$k_3 = \left(\frac{k_3}{m_5}\right) \frac{M}{\sum_{i=1}^{5} \frac{m_i}{m_5}} = 35.9150, \quad k_4 = \left(\frac{k_4}{m_5}\right) \frac{M}{\sum_{i=1}^{5} \frac{m_i}{m_5}} = 55.5319,$$

$$k_5 = \left(\frac{k_5}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 28.3384, \quad m_1 = \left(\frac{m_1}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 0.3595,$$

$$m_2 = \left(\frac{m_2}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 0.1973, \quad m_3 = \left(\frac{m_3}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 0.1254,$$

$$m_4 = \left(\frac{m_4}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 0.2433 \text{ and } m_5 = \left(\frac{m_5}{m_5}\right)\frac{M}{\sum_{i=1}^{5}\frac{m_i}{m_5}} = 0.0746.$$

The parameters k_i and m_i, for all $i=1,2,...5$ are positive and real, and the actual physical system may be reconstructed by (3.15). Substituting above values in (3.13) and (3.14), we obtain

$$\mathbf{K} = \begin{bmatrix} 135.6481 & -42.8196 & 0 & 0 & 0 \\ -42.8196 & 78.7346 & -35.9150 & 0 & 0 \\ 0 & 35.9150 & 91.4469 & -55.5319 & 0 \\ 0 & 0 & -55.5319 & 83.8703 & -28.3384 \\ 0 & 0 & 0 & -55.5319 & 28.3384 \end{bmatrix}$$

and

$$\mathbf{M} = \begin{bmatrix} 0.3595 & & & & \\ & 0.1973 & & & \\ & & 0.1254 & & \\ & & & 0.2433 & \\ & & & & 0.0746 \end{bmatrix}.$$

As mentioned in the introduction, the data needed for the reconstruction must be obtained experimentally and may be influenced by errors. In such cases the above solution may break down or may become inaccurate. In order to improve the accuracy of the finite difference model of the rod we introduce in the following section a higher order finite difference approximation.

3.3 A higher order finite difference approximation

If the *first* derivative of a finite difference model is expressed in terms of p points then the model is called a *p-point finite difference scheme.* In this definition, the second derivative of a p-point finite difference scheme is expressed in $p+1$ terms. For example, in the two point approximation $f_i' = (f_{i+1} - f_i)/h$, the second derivative of f may be expressed as

$$f_2'' = \frac{1}{h^2}(\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3), \tag{3.19}$$

where α_1, α_2 and α_3 are the parameters to be found. For simplicity, we consider the derivatives of the function $f(x)$ at the point $x=0$ (see *Figure* 3-6). Taylor's series expansion of f_1 around $x=0$ thus gives

$$f_1 = f_2 - h\hat{f}_2' + \frac{1}{2}h^2\hat{f}_2'' - \frac{1}{6}h^3\hat{f}_2''' + \frac{1}{24}h^4\hat{f}_2^{(4)} - \ldots \tag{3.20}$$

where $\hat{f}_2^{(i)}$ is the exact i-th derivative of the continuous function $f(x)$ at $x=0$. Similarly, Taylor's series expansion of f_3 around $x=0$ is

$$f_3 = f_2 + h\hat{f}_2' + \frac{1}{2}h^2\hat{f}_2'' + \frac{1}{6}h^3\hat{f}_2''' + \frac{1}{24}h^4\hat{f}_2^{(4)} + \ldots \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19) yields

$$f_2'' = \frac{1}{h^2}\left[f_2(\alpha_1 + \alpha_2 + \alpha_3) + h\hat{f}_2'(-\alpha_1 + \alpha_3) + \frac{1}{2!}h^2\hat{f}_2''(\alpha_1 + \alpha_1)\right] + \frac{1}{h^2}\left[\frac{1}{3!}h^3\hat{f}_2'''(-\alpha_1 + \alpha_3) + \frac{1}{4!}h^4\hat{f}_2^{(4)}(\alpha_1 + \alpha_3)\right] + \ldots \tag{3.22}$$

Choosing α_1, α_2 and α_3 such that

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 2 \end{cases} \tag{3.23}$$

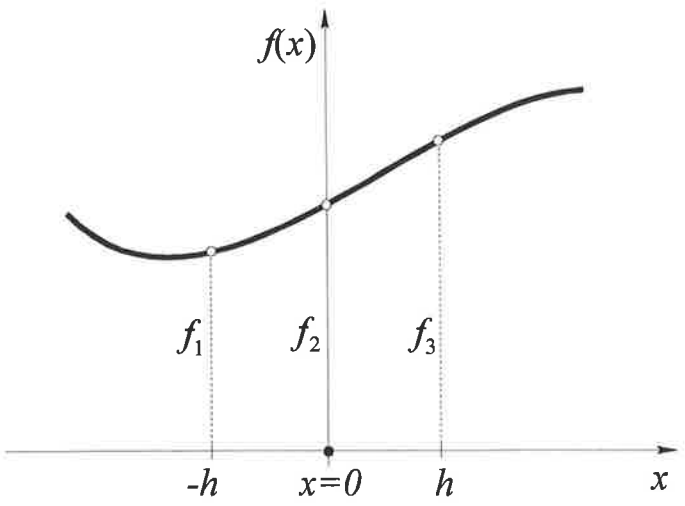

Figure 3-6 A two-point finite difference approximation for $f(x)$

we find that equation (3.22) reduces to

$$f_2''-\hat{f}_2''=\frac{2}{4!}h^2\hat{f}_2^{(4)}+O(h^3)\,. \tag{3.24}$$

The system (3.23) yields

$$\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = 1, \tag{3.25}$$

and hence the two-point finite difference approximation for f'' is given by

$$f_2''=\frac{1}{h^2}(f_1-2f_2+f_3)+O(h^2)\,, \tag{3.26}$$

where $O(h^2)$ includes terms of order h^2 and higher.

The above analysis may be applied to any other finite difference schemes. For example, in the case of the four-point approximation, we write

$$f''(x)=\frac{1}{h^2}\left[a_1f(x-2h)+a_2f(x-h)+a_3f(x)+a_4f(x+h)+a_5f(x+2h)\right], \tag{3.28}$$

where a_i, $i=1,2,\ldots,5$ are parameters to be found. As in the previous case a_i may be determined by expanding $f(x\text{-}2h)$, $f(x\text{-}h)$, $f(x+h)$ and $f(x+2h)$ around x using Taylor's series expansion, ie.

$$f(x+h)=\sum_{i=0}^{\infty}\frac{(h)^i}{i!}\frac{d^{(i)}\hat{f}(x)}{dx^i}, \tag{3.29a}$$

$$f(x+2h)=\sum_{i=0}^{\infty}\frac{(2h)^i}{i!}\frac{d^{(i)}\hat{f}(x)}{dx^i}, \tag{3.29b}$$

$$f(x-h)=\sum_{i=0}^{\infty}(-1)^i\frac{(h)^i}{i!}\frac{d^{(i)}\hat{f}(x)}{dx^i}, \tag{3.29c}$$

$$f(x-2h)=\sum_{i=0}^{\infty}(-1)^i\frac{(2h)^i}{i!}\frac{d^{(i)}\hat{f}(x)}{dx^i}. \tag{3.29d}$$

Substituting (3.29) into (3.28) yields

$$f''(x)=\frac{1}{h^2}[f(x)(a_1+a_2+a_3+a_4+a_5)]$$

$$+\frac{1}{h^2}\left[\frac{h}{1!}\hat{f}'(x)(-2a_1-a_2+a_4+2a_5)\right]$$

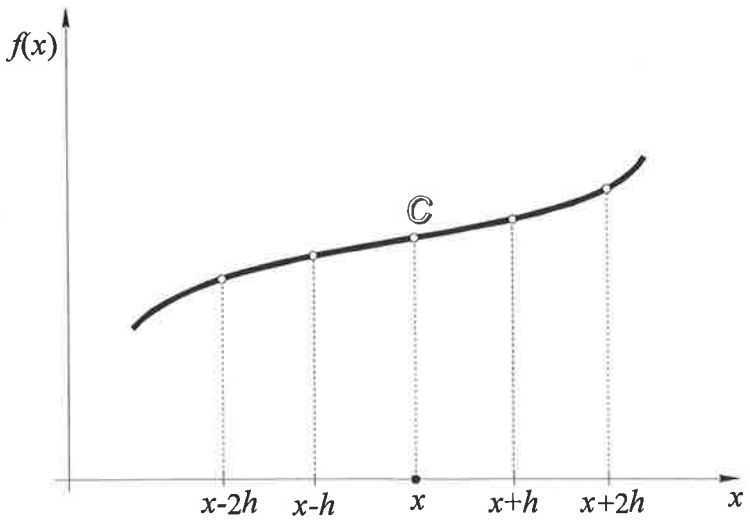

Figure 3-7 The four-point approximation for the derivatives of $f(x)$

$$+\frac{1}{h^2}\left[\frac{h^2}{2!}\hat{f}''(x)(4a_1+a_2+a_4+4a_5)\right]$$

$$+\frac{1}{h^2}\left[\frac{h^3}{3!}\hat{f}'''(x)(-8a_1-a_2+a_4+8a_5)\right] \tag{3.30}$$

$$+\frac{1}{h^2}\left[\frac{h^4}{4!}\hat{f}^{(4)}(x)(16a_1+a_2+a_4+16a_5)\right]$$

$$+\frac{1}{h^2}\left[\frac{h^5}{5!}\hat{f}^{(5)}(x)(-32a_1-a_2+a_4+32a_5)\right]$$

$$+\frac{1}{h^2}\left[\frac{h^6}{6!}\hat{f}^{(6)}(x)(64a_1+a_2+a_4+64a_5)\right]+\ldots$$

Choosing α_1, α_2, α_3, α_4 and α_5 such that

$$\begin{cases} a_1+a_2+a_3+a_4+a_5=0 \\ -2a_1-a_2+a_4+2a_5=0 \\ 4a_1+a_2+a_4+4a_5=2 \\ -8a_1-a_2+a_4+8a_5=0 \\ 16a_1+a_2+a_4+16a_5=0 \end{cases} \tag{3.31}$$

equation (3.30) reduces to

$$f''(x) - \hat{f}''(x) = \frac{h^4}{6!}\hat{f}^{(6)}\left[64a_1 + a_2 + a_4 + 64a_5\right] + \ldots \tag{3.32}$$

The system (3.31) yields

$$a_1 = -1/12,\ a_2 = 16/12,\ a_3 = -30/12, \tag{3.33a}$$

$$a_4 = 16/12,\ a_5 = -1/12. \tag{3.33b}$$

and hence (3.32) gives

$$f''(x) - \hat{f}''(x) = \frac{h^4}{90}\hat{f}^{(6)} + O(h^5). \tag{3.34}$$

The four-point finite difference approximation for the second derivative of $f(x)$ is therefore given by

$$\frac{d^2 f(x)}{dx^2} = \frac{-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h)}{12h^2} + O(h^4), \tag{3.35}$$

Similarly, it may be found that the first derivative of $f(x)$ may be approximated by

$$\frac{df(x)}{dx} = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} + O(h^4). \tag{3.36}$$

In order to demonstrate the accuracy of the above approximation we give the following numerical example.

3.3.1 Example 3-2

Consider the second derivatives of the function $f(x)=3\sin(x)$. Using equations (3.8b) and (3.35) and a step size of $h_1=0.4$ and $h_2=0.1$, the two and the four-point finite difference approximations for $f''(x)$ are calculated for several values of x and then compared with the corresponding exact values. The results are given in *Table* 3-1 and *Table* 3-2.

i	EXACT	FINITE DIFF. TWO-POINT	RELATIVE ERROR (%)	FINITE DIFF. FOUR-POINT	RELATIVE ERROR (%)
1	-2.5244	-2.4909	1.3262	-2.5237	0.0280
2	-2.7279	-2.6917	1.3262	-2.7271	0.0280
3	-0.4234	-0.4177	1.3262	-0.4232	0.0280
4	2.2704	2.2403	1.3262	2.2698	0.0280
5	2.8768	2.8386	1.3262	-2.8760	0.0280

Table 3-1. Second derivatives for the function $f(x)=3\sin(x)$ with $h=0.4$

i	EXACT	FINITE DIFF. TWO-POINT	RELATIVE ERROR (%)	FINITE DIFF. FOUR-POINT	RELATIVE ERROR (%)
1	-2.5244	-2.5223	0.0833	-2.5244	0.0000
2	-2.7279	-2.7256	0.0833	-2.7279	0.0000
3	-0.4234	-0.4230	0.0833	-0.4234	0.0000
4	2.2704	2.2685	0.0833	2.2704	0.0000
5	2.8768	2.8744	0.0833	2.8768	0.0000

Table 3-2. Second derivatives for the function $f(x)=3\sin(x)$ with $h=0.1$

As expected the results obtained by the four-point finite difference approximation are considerably more accurate then the results produced by the two-point scheme.

3.4 A higher order finite difference model of a longitudinally vibrating rod

The eigenvalue problem for the continuous model of a longitudinally vibrating rod of length L, fixed at $x=0$ and free at $x=L$, is given by (3.7). As in the standard two-point model we imagine that the system is subdivided into N equal segments, each of length $h=\frac{L}{N}$, such that the segment mass is concentrated in the middle of the elements (see *Figure* 3-8). At the points corresponding to $x=h/2$ and $x=L-h/2$ we set $i=1$ and $i=n$, respectively. Introducing

$$p(x) = \frac{E(x)A(x)}{h} \quad \text{and} \quad m(x) = \rho(x)A(x)h, \tag{3.37}$$

the eigenvalue equation (3.7) takes the form

$$h\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + \frac{1}{h}\lambda m(x)u(x) = 0, \tag{3.38}$$

or

$$h^2\left(p'(x)u'(x) + p(x)u''(x)\right) + \lambda m(x)u(x) = 0, \tag{3.39}$$

where primes denote derivatives with respect to x. In the finite difference form, the functions $p(x)$ and $m(x)$ are sampled as

$$p_i = \frac{(EA)_i}{h}, \text{and } m_i = (\rho A)_i h, \quad i = 1, 2, ..., n, \tag{3.40}$$

where E_i, A_i and ρ_i are the Young's modulus of elasticity, cross sectional area and the density of the i-th segment, respectively. Applying the four-point finite difference scheme (3.35)-(3.36) in the equation (3.39) we obtain

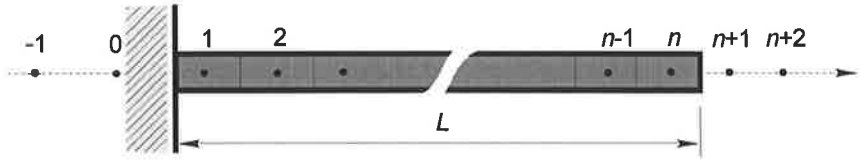

Figure 3-8 A higher order finite difference model of a vibrating rod

$$\left(\frac{h}{12}\right)^2(p_{i-2}-8p_{i-1}-12p_i+p_{i+1}-p_{i+2})$$

$$+8\left(\frac{h}{12}\right)^2(-p_{i-2}+8p_{i-1}+24p_i-8p_{i+1}+p_{i+2})u_{i-1} \tag{3.41}$$

$$+12\left(\frac{h}{12}\right)^2(30p_{i-1})u_i+8\left(\frac{h}{12}\right)^2(p_{i-2}-8p_{i-1}+24p_i+8p_{i+1}-p_{i+2})u_{i+1}$$

$$+\left(\frac{h}{12}\right)^2(-p_{i-2}+8p_{i-1}-12p_i-8p_{i+1}+p_{i+2})u_{i+2}+\lambda m_i u_i=0.$$

The above algebraic system must be satisfied for all $i=1,2,\ldots,n$. However, for the interior nodal points $i=1$ and $i=2$ at the left, and at $i=n-1$ and $i=n$ at the right, this model requires the knowledge of u_i and p_i at the nodal points $i=-1$, 0, $n+1$ and $n+2$. As these points lie outside of the physical domain of the rod, they do not have clear physical meaning and must be determined in some way. We may use here boundary conditions specified in (3.7), but they are obviously not sufficient for the determination of eight unknown p_i and m_i. Alternatively, we may apply the lower order, two-point schemes at the end points. In his case only two additional unknowns will be introduced and two equations resulted from the boundary conditions will be sufficient for their determination. It has been shown however, that the use of the lower accurate approximation at the end points decreases the accuracy not only at these points, but also at all other points of the model. For this reason we must use the same order schemes at all points and solve

the problem using some additional conditions on the data. These conditions are obtained as follows.

The quantities u_i in the finite difference model are the discrete representations of the eigenfunction $u(x)$. Because of its physical meaning, this function must be smooth and continuous inside the physical domain of the system. According to boundary conditions in (3.7), the shapes of $u(x)$ at the end points must be similar to those shown in *Figure* 3-9 and *Figure* 3-10. Geometrically, this means that $u(x)$ should be

- *anti-symmetric* about the point $x=0$, since $u(x)=0$ at $x=0$, and
- *symmetric* about the point $x=L$, since $\partial u(x)/\partial x=0$ at $x=L$.

Assuming that the function $p(x)$ is *symmetric* about the end points we obtain the following boundary conditions of the model

$$i=0: \quad \begin{cases} u_0 = -u_1 \\ \\ p_0 = p_1 \end{cases}, \tag{3.42a}$$

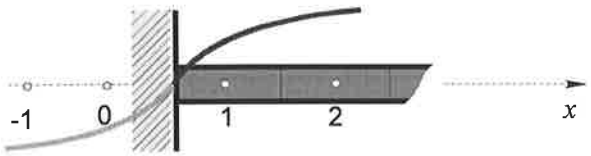

Figure 3-9 The boundary conditions at the fixed end of the rod

$$i = -1: \quad \begin{cases} u_{-1} = -u_2 \\ \\ p_{-1} = p_2 \end{cases}, \tag{3.42b}$$

$$i = n+1: \quad \begin{cases} u_{n+1} = u_n \\ \\ p_{n+1} = p_n \end{cases} \tag{3.42c}$$

and

$$i = n+2: \quad \begin{cases} u_{n+2} = u_{n-1} \\ \\ p_{n+2} = p_{n-1} \end{cases}. \tag{3.42d}$$

Introducing (3.42) into the system (3.41) and writing it in the matrix form we obtain

$$\mathbf{Ku} = \lambda \mathbf{Mu}, \tag{3.43}$$

where the stiffness and mass matrices are given by

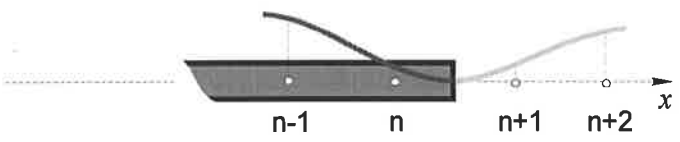

Figure 3-10 The boundary conditions at the free end of the rod

$$\mathbf{K} = \frac{n}{144L}\begin{bmatrix} k_{11} & k_{12} & k_{13} & & & & & & & \\ k_{21} & k_{22} & k_{23} & k_{24} & & & & & & \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & & & & & \\ & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & k_{i,i-2} & k_{i,i-1} & k_{i,i} & k_{i,i+1} & k_{i,i+2} & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & k_{n-2,n-4} & k_{n-2,n-3} & k_{n-2,n-2} & k_{n-2,n-1} & k_{n-2,n} \\ & & & & & & k_{n-1,n-3} & k_{n-1,n-2} & k_{n-1,n-1} & k_{n-1,n} \\ & & & & & & & k_{n,n-2} & k_{n,n-1} & k_{n,n} \end{bmatrix} \tag{3.44}$$

and

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{n-1} & \\ & & & & m_n \end{bmatrix}. \tag{3.45}$$

The parameters k_{ij} and m_{ij} in (3.44) and (3.45) are given in the Appendix 1.

The stiffness matrix **K** is now five-diagonal and generally non-symmetric. Although this non-standard form may produce certain difficulties in numerical computations for the eigenvalues and eigenvectors in the direct problem, it will shown latter that the inverse problem for this model may be solved.

In the following example we demonstrate the accuracy of the higher order model on the case of a uniform vibrating rod.

3.4.1 Example 3-3

Consider the longitudinal vibration of a fixed-free rod with constant cross-sectional area and density. Differential equation (3.7) may be solved analytically for this case. The resulting eigenvalues and their corresponding eigenfunctions are given by

$$\lambda_i = \left[\frac{(2i-1)\pi}{2L}\right]^2 \text{ and } u_i = \sin\left[\frac{(2i-1)\pi}{2L}x\right], \text{ where } i = 1, 2, \ldots.$$

In the case that the rod is approximated using the higher order finite difference schemes, the analytical model is given by (3.43). Let the length of the system be $L=1$, and assume for simplicity that $k_i = m_i = 1$ and $n=7$. Using (3.44) and (3.45) we obtain

$$\mathbf{K} = \begin{bmatrix} 187.833 & 69.4167 & 4.0833 & 0 & 0 & 0 & 0 \\ 69.4167 & 122.5000 & -65.3333 & 4.0833 & 0 & 0 & 0 \\ 4.0833 & -65.3333 & 122.5000 & -65.3333 & 4.0833 & 0 & 0 \\ 0 & 4.0833 & -65.3333 & 122.5000 & -65.3333 & 4.0833 & 0 \\ 0 & 0 & 4.0833 & -65.3333 & 122.5000 & -65.3333 & 4.0833 \\ 0 & 0 & 0 & 4.0833 & -65.3333 & 122.5000 & -61.2500 \\ 0 & 0 & 0 & 0 & 4.0833 & -61.2500 & 57.1667 \end{bmatrix},$$

and

$$\mathbf{M} = diag(1\ 1\ \ldots 1).$$

The eigenvalue problem (3.43), where **K** and **M** are given above may be solved by using standard numerical algorithms. In order to demonstrate the accuracy of the

result, the eigenvalue problem based on the two-point approximation is also solved. Table 3-3 shows numerical results for the eigenvalues obtained by the two models. As expected, the higher order solution is more accurate with respect to all modes.

MODE	EXACT	FINITE DIFF. TWO-POINT	RELATIVE ERROR (%)	FINITE DIFF. FOUR-POINT	RELATIVE ERROR (%)
1	2.4674	2.1415	13.2	2.4673	0.0028
2	22.2066	18.7163	15.7	22.1579	0.2
3	61.6850	49.0000	20.6	60.7140	1.6
4	120.9027	87.7562	27.4	114.3333	5.4
5	199.8595	128.2837	35.8	174.1023	12.9
6	298.5555	162.5748	45.5	226.4766	24.1
7	416.9908	187.5275	55.0	257.2485	38.3

Table 3-3. Eigenvalues for a uniform rod with $L=1$, $n=7$, and $k_i = m_i = 1$

3.5 An inverse problem for a higher order finite difference model of the rod

3.5.1 Introduction

An inverse problem for a longitudinally vibrating rod, based on the finite element approximation has been analysed in Ram and Gladwell [71]. It has been found there that the mass and stiffness matrices may be reconstructed from one eigenvalue, two eigenvectors and the total mass of the rod. In this model, the stiffness and mass matrices appearing in the eigenvalue problem $\mathbf{Ku}=\lambda\mathbf{Mu}$ are both tridiagonal. The last row of this system has therefore just two unknowns, the stiffness parameter k_n and the mass parameter m_n. The use of two eigenpairs $(\mathbf{u}^{(1)}, \lambda_1)$ and $(\mathbf{u}^{(2)}, \lambda_2)$ yields two matrix equations and their last rows may be used for the determination of the ratio k_n/m_n and the value of λ_2.

In the case of the four point finite difference scheme *one eigenpair, one additional eigenvector and the total mass* of the model may also be used to reconstruct the physical parameters of the system. The stiffness matrix, however, is five diagonal in this model and the two last rows of the eigenvalue problem include four unknowns: the stiffnesses k_n, k_{n-1} and k_{n-2} and the mass m_n. Two equations corresponding to the two last rows of the eigenvalue problem are now not sufficient for their determination and we must look for an alternative approach.

3.5.2 The reconstruction procedure

The eigenvalue problem of the higher order finite difference model of the rod is given by (3.44), where the stiffness and mass matrices are determined using (3.45) and (3.46). Suppose that *one eigenvalue* λ, *two eigenvectors* $\mathbf{u}$ and $\mathbf{u}^*$ and *the total mass* of the rod are given. The problem is to determine the physical parameters k_i and m_i, $i=1, 2, ...,n$ of the model.

The system (3.45) must be satisfied for both eigenpairs, which gives

$$\begin{cases} \mathbf{Ku} - \lambda\mathbf{Mu} = \mathbf{o} \\ \\ \mathbf{Ku}^* - \lambda^*\mathbf{Mu}^* = \mathbf{o} \end{cases}, \tag{3.46}$$

where λ^* is an eigenvalue of the model that is associated with the eigenvector $\mathbf{u}^*$. To obtain a solution to the problem, we first rearrange the system (3.46) in the following form

$$4p_1k_1 + 9k_2 - k_3 - q_1\lambda m_1 = 0,$$

$$4p^*_1k_1 + 9k_2 - k_3 - q_1{}^*\lambda^* m_1 = 0,$$

$$-7k_1 - 12p_2k_2 + 8k_3 - k_4 - q_2\lambda m_2 = 0,$$

$$-7k_1 - 12p^*_2k_2 + 8k_3 - k_4 - q_2{}^*\lambda^* m_2 = 0,$$

$$\vdots$$

$$- k_{i-2} + 8k_{i-1} + 12p_i k_i - 8k_{i+1} + k_{i+2} - q_i \lambda m_i = 0, \tag{3.47}$$

$$- k_{i-2} + 8k_{i-1} + 12p^*_i k_i - 8k_{i+1} + k_{i+2} - q_i^* \lambda^* m_i = 0,$$

$$\vdots$$

$$- k_{n-3} + 8k_{n-2} + 12p_{n-1}k_{n-1} - 7k_n - q_{n-1}\lambda m_{n-1} = 0,$$

$$- k_{n-3} + 8k_{n-2} + 12p^*_{n-1}k_{n-1} - 7k_n - q_{n-1}^* \lambda^* m_{n-1} = 0,$$

$$-k_{n-2} + 9k_{n-1} + 4p_n k_n - q_n \lambda m_n = 0,$$

$$-k_{n-2} + 9k_{n-1} + 4p_n k_n - q_n^* \lambda^* m_n = 0,$$

where we introduced parameters p_i and p_i^* by

$$p_i = \frac{b_i}{a_i}, p_i^* = \frac{b_i^*}{a_i^*}, \; i = 1, 2, ..., n \tag{3.48a}$$

and q_i and q_i^* by

$$q_i = \frac{144u_i}{n^2 a_i}, q_i^* = \frac{144u_i^*}{n^2 a_i^*}, \; i = 1, 2, ..., n. \tag{3.48b}$$

In the above, a_i, a_i^*, b_i and a_i^*, $i = 1, 2, ..., n$ are constant parameters, which are determined by the given eigenvectors. The explicit form of a_i, a_i^*, b_i and a_i^* are given in Appendix 2. The matrix representation of the system (3.47) is

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{A}_2 & \lambda^* \mathbf{B}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{3.49}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 4p_1 & 9 & -1 & & & & & & \\ -7 & -12p_2 & -8 & -1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & -1 & 8 & 12p_i & -8 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & -1 & 8 & 12p_{n-1} & -7 \\ & & & & & & -1 & 9 & 4p_n \end{bmatrix}, \tag{3.50}$$

$$\mathbf{A}_2 = \begin{bmatrix} 4p_1^* & 9 & -1 & & & & & & \\ -7 & -12p_2^* & -8 & -1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & -1 & 8 & 12p_i^* & -8 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & -1 & 8 & 12p_{n-1}^* & -7 \\ & & & & & & -1 & 9 & 4p_n^* \end{bmatrix}, \tag{3.51}$$

$$\mathbf{B}_1 = -\lambda \begin{bmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \ddots & & \\ & & & q_{i-1} & \\ & & & & q_i \end{bmatrix}, \tag{3.52}$$

$$\mathbf{B}_2 = -\frac{144}{n^2} \begin{bmatrix} q_1^* & & & & \\ & q_2^* & & & \\ & & \ddots & & \\ & & & q_i^* & \\ & & & & q_{i-1}^* \end{bmatrix}. \tag{3.53}$$

We note that (3.49) represents a homogeneous system of $2n$ algebraic equations in which the eigenvalue λ^* and the vectors $\mathbf{k}$ and $\mathbf{m}$ are only unknowns. To obtain a non-trivial solution for $\mathbf{k}$ and $\mathbf{m}$, the coefficient matrix of the system must be singular. As in [71] this condition may be used for the determination of the unknown λ^*. To do this we first reduce the system (3.49) using the following approach. First, write (3.49) in an expanded form, ie.

$$\mathbf{A}_1 \mathbf{k} + \mathbf{B}_1 \mathbf{m} = \mathbf{o}, \tag{3.54a}$$

$$\mathbf{A}_2 \mathbf{k} + \lambda^* \mathbf{B}_2 \mathbf{m} = \mathbf{o}. \tag{3.54b}$$

Next, use (3.54a) and evaluate

$$\mathbf{m} = -[\mathbf{B}_1]^{-1}\mathbf{A}_1 \mathbf{k}, \tag{3.55}$$

providing that $\mathbf{B}_1$ is non-singular, eg. $\lambda \neq 0$, $u_i \neq 0$ and $a_i \neq 0$, $i = 1,2, \ldots n$. Then, substituting (3.55) into (3.54b) we get

$$\mathbf{A}_2 \mathbf{k} - \lambda^* \mathbf{B}_2 [\mathbf{B}_1]^{-1}\mathbf{A}_1 \mathbf{k} = \mathbf{o}. \tag{3.56}$$

Unlike (3.49), the system (3.56) is of order n and as such is more convenient for computations. To further simplify the problem, we introduce the matrices $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A} = \mathbf{A}_2$ and $\mathbf{B} = \mathbf{B}_2 [\mathbf{B}_1]^{-1}\mathbf{A}_1$. This yields

$$[\mathbf{A} - \lambda^* \mathbf{B}]\mathbf{k} = \mathbf{o}. \tag{3.57}$$

In the above equation, $\mathbf{k}$ may be regarded as an eigenvector of the matrix pencil $(\mathbf{A}-\lambda^*\mathbf{B})$. Since this system is of order n, there will be n eigenvectors $\mathbf{k}^{(i)}$ and their corresponding eigenvalues $\lambda^{*(i)}$, $i=1, 2, ..., n$, satisfying (3.57). In order to determine the right value for λ^*, we look for one particular $\mathbf{k}^{(p)}$ from these n eigenvectors that has all positive elements. All other $\mathbf{k}^{(i)}$, $i=1, 2, ..., n$, where $i\neq p$, cannot be a solution of the problem because at least one of their elements will be negative, and therefore cannot be physically acceptable. The eigenvalue $\lambda^{*(p)}$ which is associated with this particular vector $\mathbf{k}^{(p)}$ is then a required second eigenvalue λ^*.

The above procedure is easy to implement in a computer program. For a system of n degrees-of-freedom we simply form n loops and in each of these loops examine the sign of $\mathbf{k}^{(i)}$. When for some p all $k_1^{(p)}$, $k_2^{(p)}$, ..., $k_n^{(p)}$ are positive, we exit the loops and set $\lambda^{*(p)}$ associated with this particular $\mathbf{k}^{(p)}$.

Finding the unknown eigenvalue λ^* we may now solve the system (3.49) by using the knowledge of the total mass of the system. This gives an additional equation, ie.

$$M=m_1+m_2+ \ ...+ \ m_n., \tag{3.58}$$

where M is the total mass. There is a number of ways to solve this problem. One is to simply substitute one of the equations in (3.49) with the equation (3.58) and in this way obtain a non-homogeneous system that may be solved by inverting the

resulting coefficient matrix. If we replace for example the last row of (3.49) by (3.58) we will get

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{b} \end{Bmatrix}, \tag{3.59}$$

where

$$\mathbf{C}_1 = \begin{cases} \mathbf{A}_2, & \text{for the first } (n-1) \text{ rows} \\ \\ \mathbf{0}, & \text{for the last row} \end{cases} \tag{3.60a}$$

$$\mathbf{D}_1 = \begin{cases} \lambda^* \mathbf{B}_2, & \text{for the first } (n-1) \text{ rows} \\ \\ \mathbf{I}, & \text{for the last row} \end{cases} \tag{3.60b}$$

and

$$\mathbf{b} = \begin{cases} \mathbf{0}, & \text{for the first } (n-1) \text{ elements} \\ \\ M, & \text{for the last element} \end{cases} \tag{3.60c}$$

The system (3.59) has now solution

$$\begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}^{(-1)} \begin{Bmatrix} \mathbf{0} \\ \mathbf{b} \end{Bmatrix}, \tag{3.61}$$

providing that $\det\left(\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}\right) \neq 0$.

The above solution is summarised by the following algorithm.

3.5.3 Algorithm 3-1

For an n-th order model, suppose that λ, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ and M are given In order to reconstruct the physical parameters of the system we

a) determine the coefficients a_i, b_i, a_i^*, and b_i^*, as defined in Appendix 2.

b) evaluate the coefficients p_i, q_i, p_i^*, and q_i^* using (3.48a) and (3.48b).

c) Using equations (3.50), (3.51), (3.52) and (3.53) obtain matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$.

d) Define $\mathbf{A} = \mathbf{A}_2$ and evaluate $\mathbf{B} = \mathbf{B}_2[\mathbf{B}_1]^{-1}\mathbf{A}_1$.

e) Find the n eigenvalues μ_i, $i=1, 2, ..., n$ from the condition $\det(\mathbf{A}-\mu_i\mathbf{B})=0$.

f) Determine the eigenvalue λ^* from the set of eigenvalues μ_i such that $k_j^{(i)} > 0$ for all $i=1, 2, ..., n$.

g) Form the matrices $\mathbf{C}_1$ and $\mathbf{D}_1$ and the vector $\mathbf{b}$ using (3.60).

h) Solve the system (3.61) and evaluate parameters k_i and m_i, $i = 1, 2, ..., n$.

The above algorithm has been tested on numbers of numerical experiments. To demonstrate it we present the following example.

3.5.4 Example 4

For n = 5, let the following data, corresponding to a longitudinally vibrating rod, is given:

$$\lambda_1 = 3,$$

$$\mathbf{u}^{(1)T} = \{\, 0.0600 \quad 0.2000 \quad 0.3500 \quad 0.5000 \quad 0.7500 \,\},$$

$$\mathbf{u}^{(2)T} = \{\, 0.1500 \quad 0.4500 \quad 0.6000 \quad 0.4000 \quad -0.5000 \,\},$$

$$\text{and } M = 10,$$

where λ_1 is an eigenvalue, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are eigenvectors and M is the total mass of the system. In order to reconstruct the physical parameters of the system we apply the above algorithm and obtain following:

a) We first determine the coefficients: a_1=-1.5300, a_2=-1.7600, a_3=1.7100, a_4=2.6500, a_5=1.8500, b_1=2.1900, b_2=0.1200, b_3=0.1100, b_4=-1.6500, b_5=6.3500, a^*_1=-3.7500, a^*_2=-3.0500, a^*_3=0.2500, a^*_4=-7.8500, a^*_5=-7.0000, b^*_1=7.0500, b^*_2=-1.7500, b^*_3=4.0500, b^*_4=10.3500, b^*_5=-23.2000.

b) Using these values we then evaluate the coefficients p_1=-1.4314, p_2=-0.0682, p_3=0.0643, p_4=-0.6226, p_5=3.4324, q_1=-0.2259, q_2=-0.6545, q_3=1.1789, q_4=1.0868, q_5=2.3351, p^*_1= -1.8800, p^*_2=0.5738, p^*_3=16.2000, p^*_4=-1.3185, p^*_5=3.3143, q^*_1=-0.2304, q^*_2=-0.8498, q^*_3=13.8240, q^*_4=-0.2935, q^*_5=0.4114.

c) By (3.50), (3.51), (3.52) and (3.53) we determine the matrices $\mathbf{A}_i\,\mathbf{B}_i$ as

$$\mathbf{A}_1 = \begin{bmatrix} -5.7255 & 9.0000 & -1.0000 & 0.0000 & 0.0000 \\ -7.0000 & 0.8182 & 8.0000 & -1.0000 & 0.0000 \\ -1.0000 & 8.0000 & 0.7719 & -8.0000 & 1.0000 \\ 0.0000 & -1.0000 & 8.0000 & -7.4717 & -7.0000 \\ 0.0000 & 0.0000 & -1.0000 & 9.0000 & 13.7297 \end{bmatrix}, \tag{3.62a}$$

$$\mathbf{A}_2 = \begin{bmatrix} -7.5200 & 9.0000 & -1.0000 & 0.0000 & 0.0000 \\ -7.0000 & -6.8852 & 8.0000 & -1.0000 & 0.0000 \\ -1.0000 & 8.0000 & 194.4000 & -8.0000 & 1.0000 \\ 0.0000 & -1.0000 & 8.0000 & -15.8217 & -7.0000 \\ 0.0000 & 0.0000 & -1.0000 & 9.0000 & 13.2571 \end{bmatrix}, \tag{3.62b}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0.6776 & & & & \\ & 1.9636 & & & \\ & & -3.5368 & & \\ & & & -3.2604 & \\ & & & & -7.0054 \end{bmatrix}, \tag{3.63a}$$

$$\mathbf{B}_2 = \begin{bmatrix} 0.2304 & & & & \\ & 0.8498 & & & \\ & & -13.8240 & & \\ & & & 0.2935 & \\ & & & & -0.4114 \end{bmatrix}. \tag{3.63b}$$

e) The matrix $\mathbf{B}$ is determined from $\mathbf{B} = \mathbf{B}_2[\mathbf{B}_1]^{-1}\mathbf{A}_1$, which gives

$$\mathbf{B} = \begin{bmatrix} -1.9467 & 3.0600 & -0.3400 & 0 & 0 \\ -3.0295 & 0.3541 & 3.4623 & -0.4320 & -3.908 \\ -3.9086 & 31.2686 & 3.0171 & -31.2686 & 3.9086 \\ 0 & 0.0900 & -0.7202 & 0.6726 & 0.6301 \\ 0 & 0 & -0.0587 & 0.5286 & 0.8063 \end{bmatrix} \tag{3.64}$$

f) Next, we solve the eigenvalue problem $(\mathbf{A}-\mu\mathbf{B})\mathbf{k}=\mathbf{0}$ and obtain the following values for μ

$$\mu=\begin{Bmatrix} 0.6839+11.3000i \\ 0.6839-11.3000i \\ 3.5039+1.5550i \\ 3.5039-1.5550i \\ 16.7299 \end{Bmatrix} \tag{3.65}$$

and five corresponding eigenvectors $\mathbf{k}^{(j)}$, where $i = (-1)^{1/2}$. It is shown that only $\mathbf{k}^{(5)}$ has all positive elements, so we set

$$\lambda^{*} = 16.7299.$$

g) We form the matrices $\mathbf{C}_1$ and $\mathbf{D}_1$ and the vector $\mathbf{b}$, as defined in (3.60).

h) Solving the system (3.61), we obtain the unknown physical parameters k_i and m_i. The results are given in *Table* 3-4

NODE	k_i	m_i.
1	3.6500	2.0617
2	2.4698	1.5526
3	2.7274	2.3189
4	1.3388	1.3682
5	0.6980	2.6986

Table 3-4 The reconstructed physical parameters

Finding the physical parameters of the model we may now reconstruct the stiffness and mass matrices via (3.44) and (3.45), ie.

$$\mathbf{K} = \begin{bmatrix} 72.6527 & -23.4968 & 1.1841 & 0 & 0 \\ -19.0788 & 30.8728 & -15.0573 & 0.8531 & 0 \\ 1.3481 & -19.8761 & 34.0926 & -16.4893 & 0.9247 \\ 0 & 1.0600 & -12.9429 & 16.7348 & -4.8519 \\ 0 & 0 & 0.4206 & -5.5304 & 5.1098 \end{bmatrix} \tag{3.66}$$

and

$$\mathbf{M} = \text{diag}\{\ 0.4123 \quad 0.3105 \quad 0.4638 \quad 0.2736 \quad 0.5397\ \}, \tag{3.67}$$

and in this way we may reconstruct the model using (3.43).

To verify this result, the eigenvalue problem (3.43) has been solved using the above **K** and **M**. The resulting five eigenpairs $[\mu_i, \phi^{(i)}]$ are given in *Table* 3-5.

NODE	μ_1=3.0000	μ_2=16.7299	μ_3=63.1152	μ_4=123.3761	μ_5=213.5284
	$\phi^{(1)}$	$\phi^{(2)}$	$\phi^{(3)}$	$\phi^{(4)}$	$\phi^{(5)}$
1	0.0607	0.1504	0.3119	0.4065	0.8220
2	0.2022	0.4511	0.6204	0.3438	-0.5286
3	0.3538	0.6015	0.0293	-0.6547	0.1981
4	0.5054	0.4010	-0.7062	0.5341	-0.0755
5	0.7582	-0.5013	0.1353	-0.0525	0.0045

Table 3-5 The eigenvalues and their corresponding eigenvectors

Upon normalisation $\phi^{(1)} = \phi^{(1)}\frac{0.0600}{0.0607}$ and $\phi^{(2)} = \phi^{(2)}\frac{0.1500}{0.1504}$ we obtain as required

$$\phi^{(1)T} = \{\, 0.0600 \quad 0.2000 \quad 0.3500 \quad 0.5000 \quad 0.7500 \,\},$$

$$\phi^{(2)T} = \{\, 0.1500 \quad 0.4500 \quad 0.6000 \quad 0.4000 \quad -0.5000 \,\}.$$

3.5.5 Example 5

Suppose that for n=10 the following data, corresponding to the *continuous model* of a uniform rod with $k_i = m_i = 1$, $l = 1$, is given by

$\lambda_1 = 2.4674,$

$\mathbf{u}^{(1)T}$={0.0785 0.2334 0.3827 0.5225 0.6494 0.7604 0.852 0.9239 0.9724 0.9969},

$\mathbf{u}^{(2)T}$={0.2334 0.6494 0.9239 0.9969 0.8526 0.5225 0.0785 -0.3827 -0.7604 -0.9724}

and $M = 10$.

Solving the eigenvalue problem (3.57) for this case, the following values for μ_i have been found:

-118.84	37.39	24.63	22.19	-8.88	-1.14	2.54	2.29-0.0006*i*	2.29+0.0006*i*

It is shown that the only $\mathbf{k}^{(4)}$ has all positive elements, thus we set

$$\lambda^* = \mu_4 = 22.19.$$

Finding the unknown eigenvalue λ^*, we continue the reconstruction procedure as in the previous example. The results for the physical parameters are given in *Table* 3-6.

NODE	k_i	m_i	NODE	k_i	m_i
1	1.00	1.00	6	1.00	1.00
2	1.00	1.00	7	1.00	1.00
3	1.00	1.00	8	1.00	1.00
4	1.00	1.00	9	1.00	1.00
5	1.00	1.00	10	1.00	1.00

Table 3-6 The reconstructed physical parameters

We note that in this case the solution is exact.

3.6 Numerical test of sensitivity

The values for given eigenvalue and eigenvectors in the previous examples are such that the algorithm produces positive values for the parameters k_i and m_i, as it must be for a real physical system. In a practical situation the eigendata are usually obtained from experimental testings and normally include the measurement errors. It is thus important to investigate the impact of data perturbations on the proposed solution. In order to numerically test the sensitivity of the

reconstruction algorithm, we now introduce some random small errors in the given eigenvectors. Let the perturbated eigenvectors be given by

$$\mathbf{u}^{(1)\,T} = \{\,0.0653 \quad 0.2053 \quad 0.3553 \quad 0.5053 \quad 0.7553\},$$

$$\mathbf{u}^{(2)\,T} = \{\,0.1567 \quad 0.4567 \quad 0.6067 \quad 0.4067 \quad -0.4933\}.$$

Applying the algorithm 3-1 for this case resulted in the following values for the physical parameters

NODE	k_i	m_i.
1	4.3452	1.4459
2	3.6242	2.1276
3	3.1343	2.5039
4	2.2192	2.1682
5	-0.3249	1.7544

Table 3-7 The reconstructed physical parameters

Similarly, introducing small errors in the vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ from example 5, ie.

$\mathbf{u}^{(1)T}$ ={0.0985 0.2534 0.4027 0.5425 0.6694 0.7804 0.8726 0.9439 0.9924 1.0169},

$\mathbf{u}^{(2)T}$ ={0.2634 0.6794 0.9539 1.0269 0.8826 0.5525 0.1085 -0.3527 -0.7304 -0.9424},

we obtain

NODE	k_i	m_i	NODE	k_i	m_i
1	0.9853	-0.2760	6	1.3875	1.2872
2	1.3697	1.2721	7	1.0214	0.9477
3	0.9874	0.9087	8	1.4180	1.3148
4	1.3743	1.2708	9	1.0836	1.0006
5	0.9976	0.9245	10	1.5446	1.3496

Table 3-8 The reconstructed physical parameters

Clearly, not only are the values for k_i and m_i completely corrupted, but also the reconstructed system can not be physically realised since the stiffness parameter k_5 in the first case and the mass parameter m_1 in the second have now the negative values. We note that this is expected because of the sensitivity of inverse problems in general.

In order to obtain better results and decrease the sensitivity of the procedure, we may include more eigendata into analysis. Normally, for the higher-degrees-of-freedom systems, this data may be accurately obtained by the experimental testing. Suppose that s eigenpairs $(\lambda_i, \mathbf{u}^{(i)})$, $i=1, 2, ..., s$ of the model, where $s\geq 3$, are given. The eigenvalue problem (3.43) must be satisfied for all of s eigenpairs. This yields an overdetermined system, which have an exact solution only in some particular cases. However, instead of looking for an exact solution, the system may be

solved in an optimal sense, applying for example the least squares method. This solution is examined in the following section.

3.7 The least square solution

Suppose that s - eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, their s - corresponding eigenvectors $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(s)}$ and the total mass of a rod are known, where $s \geq 3$. Introducing s eigenpairs $(\lambda_i, u^{(i)})$ into the system (3.49) we obtain the following system

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{A}_2 & \mathbf{B}_2 \\ \vdots & \vdots \\ \mathbf{A}_s & \mathbf{B}_s \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{3.68}$$

where

$$\mathbf{A}_i = \begin{bmatrix} 4p_1^{(i)} & 9 & -1 & & & & & & \\ -7 & -12p_2^{(i)} & -8 & -1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & -1 & 8 & 12p_k^{(i)} & -8 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & -1 & 8 & 12p_{n-11}^{(i)} & -7 \\ & & & & & & -1 & 9 & 4p_n^{(i)} \end{bmatrix}, \tag{3.69}$$

$$\mathbf{B}_i = -\lambda_i \begin{bmatrix} q_1^{(i)} & & & \\ & q_2^{(i)} & & \\ & & \ddots & \\ & & & q_n^{(i)} \end{bmatrix}, \tag{3.70}$$

and

$$p_k^{(i)} = \frac{b_k^{(i)}}{a_k^{(i)}}, \quad q_k^{(i)} = \frac{144u_k^{(i)}}{n^2 a_k^{(i)}}, \quad k=1,2,\ldots,n \ ; \ i=1,2,\ldots,s. \tag{3.71}$$

As in the previous section, we take into an account the knowledge of the total mass of the system M and replace one of the rows of (3.68) by

$$M=m_1 + m_2 + \ldots + m_n. \tag{3.72}$$

This gives a non-homogeneous system of the form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{A}_2 & \mathbf{B}_2 \\ \vdots & \vdots \\ \mathbf{A}_{s-1} & \mathbf{B}_{s-1} \\ \mathbf{C}_s & \mathbf{D}_s \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{b} \end{Bmatrix}, \tag{3.73}$$

where

$$\mathbf{C}_s = \begin{cases} \mathbf{A}_s, & \text{for the first } (n-1) \text{ rows} \\ \mathbf{0}, & \text{for the last row} \end{cases}, \tag{3.74a}$$

$$\mathbf{D}_s = \begin{cases} \mathbf{B}_s, & \text{for the first } (n-1) \text{ rows} \\ \mathbf{I}, & \text{for the last row} \end{cases} \tag{3.74b}$$

and

$$\mathbf{b} = \begin{cases} \mathbf{0}, & \text{for the first } (n-1) \text{ elements} \\ M, & \text{for the last element} \end{cases} \qquad (3.74c)$$

The least square solution of (3.73) may be obtained by finding the Moore-Penrose pseudo-inverse of the coefficient matrix, which is a standard numerical routine.

We summarise the above procedure in the following algorithm.

3.7.1 Algorithm 3-2

Suppose that for given n, s eigenvalues, s corresponding eigenvectors and the total mass are given ($n \geq s > 2$). To reconstruct the physical parameters of the system we

a) set n, λ_i, $\mathbf{u}^{(i)}$, $i = 1, 2, \ldots, s$ and $M = \sum_{i=1}^{n} m_i$

b) Determine the coefficients a_i and b_i, as defined in Appendix I.

c) Evaluate the coefficients $p_k^{(i)} = \dfrac{b_k^{(i)}}{a_k^{(i)}}$, $q_k^{(i)} = \dfrac{144 u_k^{(i)}}{n^2 a_k^{(i)}}$, $k = 1,2, \ldots, n$; $i = 1,2,\ldots,s$.

d) Using (3.69) and (3.70) obtain matrices $\mathbf{A}_i$ and $\mathbf{B}_i$.

e) Form the matrices $\mathbf{C}_s$ and $\mathbf{D}_s$ and the vector $\mathbf{b}$, as defined in (3.74).

f) Solve the system (3.73) by least squares and evaluate parameters k_i and m_i, $i=1, 2, ..., n$.

To demonstrate the algorithm, the following example is presented.

3.7.2 Example 6

Consider again example 5 and assume that the following eigendata are given

$$\lambda_1 = 2.5, \lambda_2 = 22, \lambda_3 = 61,$$

$$\mathbf{u}^{(1)T}=\{0.0985\ 0.2534\ 0.4027\ 0.5425\ 0.6694\ 0.7804\ 0.8726\ 0.9439\ 0.9924\ 1.0169\}$$

$$\mathbf{u}^{(2)T}=\{0.2634\ 0.6794\ 0.9539\ 1.0269\ 0.8826\ 0.5525\ 0.1085\ -0.3527\ -0.7304\ -0.9424\}$$

$$\mathbf{u}^{(3)}=\{0.3927\ 0.9339\ 0.9339\ 0.3927\ -0.3727\ -0.9139\ -0.9139\ -0.3727\ 0.3927\ 0.9339\},$$

$$M = 10.$$

We now solve the problem by using the algorithm 3-2. As *Table* 3-9 indicates, the resulting values for the physical parameters are now all real and positive, so the model may be physically reconstructed. If instead of three, five eigenpairs are used, ie. if additionally

$$\lambda_4 = 120, \lambda_5 = 200,$$

$$\mathbf{u}^{(4)}=\{0.5425\ 0.0169\ 0.4027\ -0.6294\ -0.9524\ -0.2134\ 0.7804\ 0.9439\ 0.0985\ -0.8326\},$$

$\mathbf{u}^{(3)}$={0.6594 0.8626 -0.3727 -0.9624 0.0885 1.0069 0.2434 -0.9139 -0.5125 0.7704}

are given, the algorithm will produce more accurate results (see *Table* 3-9).

NODE	THREE EIGENPAIRS		FIVE EIGENPAIRS	
	k_i	m_i	k_i	m_i
1	0.9894	0.3770	0.8358	0.6912
2	1.3435	1.3690	1.0494	1.0193
3	0.9897	0.9820	0.9095	0.9007
4	1.2838	1.2606	1.0356	1.0148
5	0.9470	0.9474	0.9203	0.9322
6	1.2452	1.2564	1.0147	0.9658
7	0.9122	0.9382	0.9307	0.9298
8	1.1973	1.2554	0.9795	0.9531
9	0.8353	0.9950	0.9088	0.9048
10	1.0482	0.9471	0.9716	0.9338

Table 3-9 The reconstructed physical parameters

3.8 Summary

A solution for the inverse problem of a longitudinally vibrating rod, modelled using a higher order finite difference approximation has been presented. It has been shown that the higher order finite difference model accurately predicts the

eigenvalues and their corresponding eigenvectors for the lower modes. Since this data may be accurately obtained by experimental testings, the solution to the inverse problem may be realistic.

In the case where the given data are noise corrupted, the solution obtained by the proposed algorithm may be inaccurate or unrealistic. To solve this problem and reduce its sensitivity to data perturbations, we took more eigenpairs into an account and developed new reconstruction algorithm by using least squares.

4 An inverse problem for the general discrete model of vibratory systems

4.1 Introduction

As mentioned in the introduction, the analytical models based on distributed physical parameters are applicable only for some particular systems of simple geometry and boundary conditions. A standard approach is to approximate the real continuous models by using some of numerical methods, such as the method of finite differences or the finite elements method. In the previous chapter we used the method of finite differences and applied a higher order scheme in the differential equation of a longitudinally vibrating rod. The improved accuracy of the higher order model is used in the inverse approach, ie. in the problem of reconstructing the physical parameters from given eigendata.

This chapter develops the reconstruction procedure for the general discrete model of linear one-dimensional vibrating systems. This may be a finite difference, a finite element, or a lumped-mass model of a vibrating system. We first develop the solution for the general case and then we apply the solution on some particular cases.

4.2 The general case

In the case of a discrete undamped linear model, the differential equations that describe the free vibration may be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{o}, \tag{4.1}$$

where **M** and **K** are the mass and stiffness matrices and **q** is the vector of displacements. Assuming harmonic response

$$\mathbf{q} = \mathbf{u}\sin \omega t, \tag{4.2}$$

system (4.1) yields the matrix eigenvalue problem

$$\mathbf{K}\mathbf{u} - \lambda\mathbf{M}\mathbf{u} = \mathbf{o}, \tag{4.3}$$

where $\lambda_i = \omega_i^2$, $i=1, 2, \ldots n$ and $\mathbf{u}^{(i)}$ are the eigenvalues and their corresponding eigenvectors, respectively.

By this model, the physical parameters of the original continuous system are discretized for each subdomain of the model. In equation (4.3) **K** and **M** are the *global* stiffness and mass matrices, assembled from the corresponding element matrices following the schemes that are known from the actual discrete model. In general, the stiffness and mass matrices for an element (e) may be expressed as the functions $\mathbf{K}_{(e)}=\mathbf{K}_{(e)}(\alpha, \alpha_2, \ldots, \alpha_n)$ and $\mathbf{M}_{(e)}=\mathbf{M}_{(e)}(\beta_1, \beta_2, \ldots, \beta_n)$, where $\alpha, \alpha_2, \ldots,$

α_n and β_1, β_2, ..., β_n are structural (physical) properties of the system. Analytically, this may be written as

$$\mathbf{K}_{(e)} = \alpha_{(e)}\mathbf{G}_{(e)} \tag{4.4}$$

and

$$\mathbf{M}_{(e)} = \beta_{(e)}\mathbf{Y}_{(e)}, \tag{4.5}$$

where

$\alpha_{(e)}$ and $\beta_{(e)}$ represent the physical parameters of the element (e), and

$\mathbf{G}_{(e)}$ and $\mathbf{Y}_{(e)}$ are known matrices determined by the actual analytical model.

For example, in the case of *the finite element model of a vibrating rod*, the coefficients $\alpha_{(e)}$ and $\beta_{(e)}$ are given by

$$\alpha_{(e)} = \frac{(EA)_{(e)}}{h} \quad \text{and} \quad \beta_{(e)} = \frac{(\rho A)_{(e)}h}{6},$$

and element matrices $\mathbf{G}_{(e)}$ and $\mathbf{Y}_{(e)}$ by

$$\mathbf{G}_{(e)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{Y}_{(e)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

where $E_{(e)}$ is the Young's modulus of elasticity, $A_{(e)}$ is the cross sectional area, $\rho_{(e)}$ is the density and h is the length of the element (e).

The overall stiffness and mass matrices for the whole system may be obtained by assembling the element matrices, ie.

$$\mathbf{K} = \sum_{e=1}^{N} \mathbf{A}_e^T \mathbf{K}_{(e)} \mathbf{A}_e \tag{4.6}$$

and

$$\mathbf{M} = \sum_{e=1}^{N} \mathbf{A}_e^T \mathbf{M}_{(e)} \mathbf{A}_e , \tag{4.7}$$

In the above, $\mathbf{A}_e$ are Boolean mapping matrices, and N is the number of elements of the model. For the finite element model of the axially vibrating rod for example, these matrices are given by

$$\mathbf{K} = \sum_{e=1}^{N} \mathbf{A}_e^T \frac{(EA)_{(e)}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{A}_e$$

and

$$\mathbf{M} = \sum_{e=1}^{N} \mathbf{A}_e^T \frac{h(\rho A)_{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{A}_e ,$$

where

$$\mathbf{A}_i = \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}}^{N}, \quad a_{jk} = \begin{cases} 1; & j = 1, k = i \\ 1; & j = 2, k = i+1 \\ 0, & \textit{elsewhere} \end{cases}.$$

Let us introduce the matrices $\mathbf{P}_{(e)}$ and $\mathbf{Q}_{(e)}$, defined as

$$\mathbf{P}_{(e)} = \mathbf{A}_e^T \mathbf{G}_{(e)} \mathbf{A}_e \tag{4.8}$$

and

$$\mathbf{Q}_{(e)} = \mathbf{A}_e^T \mathbf{Y}_{(e)} \mathbf{A}_e . \tag{4.9}$$

Then, using (4.8) and (4.9), the expressions for the stiffness and mass matrices may be simplified as

$$\mathbf{K} = \sum_{e=1}^{N} \alpha_{(e)} \mathbf{P}_{(e)} \tag{4.10}$$

and

$$\mathbf{M} = \sum_{e=1}^{N} \beta_{(e)} \mathbf{Q}_{(e)} . \tag{4.11}$$

Hence, the original eigenvalue problem (4.3) transforms to

$$\sum_{e=1}^{N} \alpha_{(e)} \mathbf{P}_{(e)} \mathbf{u} - \lambda \sum_{e=1}^{N} \beta_{(e)} \mathbf{Q}_{(e)} \mathbf{u} = \mathbf{o}. \tag{4.12}$$

The matrices $\mathbf{P}_{(e)}$ and $\mathbf{Q}_{(e)}$ are N by N matrices and $\mathbf{u}$ is a N dimensional vector. Denote

$$\mathbf{a}_{(e)} = \mathbf{P}_{(e)} \mathbf{u} \tag{4.13}$$

and

$$\mathbf{b}_{(e)} = \mathbf{Q}_{(e)}\,\mathbf{u}. \tag{4.14}$$

Then using (4.13) and (4.14) the system (4.12) may be written as

$$\sum_{e=1}^{N}\alpha_{(e)}\mathbf{a}_{(e)} - \lambda\sum_{e=1}^{N}\beta_{(e)}\mathbf{b}_{(e)} = \mathbf{o}. \tag{4.15}$$

4.3 The inverse problem for the general case

The analytical model of the system is given by (4.15). We state the following problem for this model:

> *Given one eigenvalue, two eigenvectors and one of the physical parameters of the system (ie. the total mass). Determine the physical parameters* $\alpha_{(e)}, \beta_{(e)}, e = 1,2,\ldots,N$ *that correspond to this data.*

The eigenvalue equation (4.15) must be satisfied for all eigenpairs that $(\lambda_i,\mathbf{u}^{(i)})$, $i=1,2,\ldots,n$. Suppose that $(\lambda_1,\mathbf{u}^{(1)})$ and $(\lambda_2,\mathbf{u}^{(2)})$ are two eigenpairs that correspond to model. Then, (4.15) yields

$$\sum_{e=1}^{N}\alpha_{(e)}\mathbf{a}_{(e)}^{(1)} - \lambda_1\sum_{e=1}^{N}\beta_{(e)}\mathbf{b}_{(e)}^{(1)} = \mathbf{o} \tag{4.16}$$

and

$$\sum_{e=1}^{N} \alpha_{(e)} \mathbf{a}_{(e)}^{(2)} - \lambda_2 \sum_{e=1}^{N} \beta_{(e)} \mathbf{b}_{(e)}^{(2)} = \mathbf{o}, \tag{4.17}$$

where we introduced

$$\mathbf{a}_{(e)}^{(1)} = \mathbf{P}_{(e)} \mathbf{u}^{(1)}, \tag{4.18}$$

$$\mathbf{b}_{(e)}^{(1)} = \mathbf{Q}_{(e)} \mathbf{u}^{(1)}, \tag{4.19}$$

and

$$\mathbf{a}_{(e)}^{(2)} = \mathbf{P}_{(e)} \mathbf{u}^{(2)}, \tag{4.20}$$

$$\mathbf{b}_{(e)}^{(2)} = \mathbf{Q}_{(e)} \mathbf{u}^{(2)}, \tag{4.21}$$

We first write (4.17) and (4.18) in the form

$$\left[\mathbf{a}_1^{(1)} \quad \mathbf{a}_2^{(1)} \quad \dots \quad \mathbf{a}_N^{(1)}\right] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{Bmatrix} - \lambda_1 \left[\mathbf{b}_1^{(1)} \quad \mathbf{b}_2^{(1)} \quad \dots \quad \mathbf{b}_N^{(1)}\right] \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{4.22}$$

$$\left[\mathbf{a}_1^{(2)} \quad \mathbf{a}_2^{(2)} \quad \dots \quad \mathbf{a}_N^{(2)}\right] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{Bmatrix} - \lambda_2 \left[\mathbf{b}_1^{(2)} \quad \mathbf{b}_2^{(2)} \quad \dots \quad \mathbf{b}_N^{(2)}\right] \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \tag{4.23}$$

where the physical parameters are presented as vectors. Then, we introduce the N x N matrices $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ by

$$\mathbf{A}^{(1)} = \left[\mathbf{a}_1^{(1)} \quad \mathbf{a}_2^{(1)} \quad \dots \quad \mathbf{a}_N^{(1)}\right], \tag{4.24a}$$

$$\mathbf{B}^{(1)} = \left[\mathbf{b}_1^{(1)} \quad \mathbf{b}_2^{(1)} \quad \dots \quad \mathbf{b}_N^{(1)}\right], \tag{4.24b}$$

$$\mathbf{A}^{(2)} = \left[\mathbf{a}_1^{(2)} \quad \mathbf{a}_2^{(2)} \quad \dots \quad \mathbf{a}_N^{(2)}\right], \tag{4.24c}$$

$$\mathbf{B}^{(2)} = \left[\mathbf{b}_1^{(2)} \quad \mathbf{b}_2^{(2)} \quad \dots \quad \mathbf{b}_N^{(2)}\right]. \tag{4.24d}$$

and write (4.22) and (4.23) as

$$\mathbf{A}^{(1)}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{Bmatrix} - \lambda_1\mathbf{B}^{(1)}\begin{Bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ \vdots\\ 0\end{Bmatrix}, \tag{4.26}$$

$$\mathbf{A}^{(2)}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{Bmatrix} - \lambda_2\mathbf{B}^{(2)}\begin{Bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ \vdots\\ 0\end{Bmatrix}. \tag{4.27}$$

The above two systems may be then combined into a single matrix system, ie.

$$\begin{bmatrix}\mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)}\\ \mathbf{A}^{(2)} & -\lambda_2\mathbf{B}^{(2)}\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ \vdots\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{Bmatrix}. \tag{4.28}$$

The system (4.28) is a homogeneous system of $2N$ equations. The trivial solution $\alpha_i=0$ and $\beta_i=0$ is not of interest. The non-trivial solution is associated with the conditions

$$\det\begin{bmatrix} \mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)} \\ \mathbf{A}^{(2)} & -\lambda_2\mathbf{B}^{(2)} \end{bmatrix} = 0. \tag{4.29}$$

As in the previous chapter, this condition may be used to evaluate the unknown eigenvalue λ_2, since λ_1 is given and the matrices $\mathbf{A}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{B}^{(2)}$ are known from (4.25a)-(4.25d). In order to do this we write (4.28) in the form

$$\left(\begin{bmatrix} \mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)} \\ \mathbf{A}^{(2)} & \mathbf{0} \end{bmatrix} - \lambda_2\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{(2)} \end{bmatrix}\right)\begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{4.30}$$

where we introduced vectors $\mathbf{k}$ and $\mathbf{m}$ as

$$\mathbf{k} = \{\alpha_1, \alpha_2, ..., \alpha_N\}^T \tag{4.31a}$$

and

$$\mathbf{m} = \{\beta_1, \beta_2, ..., \beta_N\}^T. \tag{4.31b}$$

System (4.30) has non-trivial solution if

$$\det\left(\begin{bmatrix}\mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)}\\ \mathbf{A}^{(2)} & \mathbf{0}\end{bmatrix}-\lambda_2\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^{(2)}\end{bmatrix}\right)=0. \tag{4.34}$$

We note from (4.34) that λ_2 may be considered as an eigenvalue of the generalised pencil $\begin{bmatrix}\mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)}\\ \mathbf{A}^{(2)} & \mathbf{0}\end{bmatrix}-\lambda_2\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^{(2)}\end{bmatrix}$. However, this eigenvalue problem is indefinite, because $\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^{(2)}\end{bmatrix}$ is singular and $\begin{bmatrix}\mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)}\\ \mathbf{A}^{(2)} & \mathbf{0}\end{bmatrix}$ is neither semi-positive definite nor negative-definite. Such eigenvalue problem is numerically difficult to solve as it includes N unbounded eigenvalues. In order to solve this problem we write (4.30) as

$$\mathbf{A}^{(1)}\mathbf{k}-\lambda_1\mathbf{B}^{(1)}\mathbf{m}=\mathbf{o} \tag{4.35}$$

and

$$\mathbf{A}^{(2)}\mathbf{k}-\lambda_2\mathbf{B}^{(2)}\mathbf{m}=\mathbf{o}. \tag{4.36}$$

Then, from (4.35) we evaluate the vector $\mathbf{m}$, ie. $\mathbf{m}=[\lambda_1\mathbf{B}^{(1)}]^{-1}\mathbf{A}^{(1)}\mathbf{a}$ and substitute the result in (4.36). This gives

$$\{\mathbf{A}^{(2)}-\lambda_2\mathbf{B}^{(2)}[\lambda_1\mathbf{B}^{(1)}]^{-1}\mathbf{A}^{(1)}\}\mathbf{k}=\mathbf{o}, \tag{4.37}$$

or

$$[\mathbf{A}^{(2)} - \lambda_2 \mathbf{C}]\,\mathbf{k} = 0, \tag{4.38}$$

where the matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B}^{(2)}[\,\lambda_1 \mathbf{B}^{(1)}\,]^{-1}\mathbf{A}^{(1)}. \tag{4.39}$$

The difference between (4.38) and the original system is that neither $\mathbf{A}^{(2)}$ nor $\mathbf{C}$ are singular, so the condition

$$\det([\mathbf{A}^{(2)} - \lambda_2 \mathbf{C}]) = 0 \tag{4.40}$$

may be solved by the standard numerical algorithms. As a result, the N eigenvalues $\lambda_2^{(1)}$, $\lambda_2^{(2)}$, ..., $\lambda_2^{(N)}$ and their corresponding eigenvectors $\mathbf{k}^{(1)}$, $\mathbf{k}^{(2)}$, ..., $\mathbf{k}^{(N)}$ will be obtained. It is shown however, that only one particular vector, say $\mathbf{k}^{(p)}$ has all positive elements, while at least one of the elements of all other $\mathbf{k}^{(i)}$ will be negative. Since by definition $\mathbf{k}$ represents the physical parameters α_i, this means that this particular $\mathbf{k}^{(p)}$ is the only eigenvector of (4.38) corresponding to the physical system. The eigenvalue $\lambda_2^{(p)}$ that is associated with $\mathbf{k}^{(p)}$ is thus the required second eigenvalue of the model.

Alternatively, in the case of the finite difference or the finite element models, where the matrices $\mathbf{A}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{B}^{(2)}$ are of an upper-triangular form, ie.

$$\mathbf{A}^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1N}^{(1)} \\ & a_{21}^{(1)} & \cdots & a_{2N}^{(1)} \\ & & \ddots & \vdots \\ & & & a_{NN}^{(1)} \end{bmatrix}, \quad \mathbf{A}^{(2)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1N}^{(2)} \\ & a_{21}^{(2)} & \cdots & a_{2N}^{(2)} \\ & & \ddots & \vdots \\ & & & a_{NN}^{(2)} \end{bmatrix}, \tag{4.41a}$$

and

$$\mathbf{B}^{(1)} = \begin{bmatrix} b_{11}^{(1)} & b_{12}^{(1)} & \cdots & b_{1N}^{(1)} \\ & b_{21}^{(1)} & \cdots & b_{2N}^{(1)} \\ & & \ddots & \vdots \\ & & & b_{NN}^{(1)} \end{bmatrix}, \quad \mathbf{B}^{(2)} = \begin{bmatrix} b_{11}^{(2)} & b_{12}^{(2)} & \cdots & b_{1N}^{(2)} \\ & b_{21}^{(2)} & \cdots & b_{2N}^{(2)} \\ & & \ddots & \vdots \\ & & & b_{NN}^{(2)} \end{bmatrix} \tag{4.41b}$$

we may determine λ_2 from the last rows of (4.35) and (4.36). In this case, these two equations are given by

$$a_{NN}^{(1)}\alpha_N - \lambda_1 b_{NN}^{(1)}\beta_N = 0 \tag{4.42a}$$

and

$$a_{NN}^{(2)}\alpha_N - \lambda_2 b_{NN}^{(2)}\beta_N = 0. \tag{4.42b}$$

The eigenvalue λ_2 may be now determine from

$$\det\left(\begin{bmatrix} a_{NN}^{(1)} & -\lambda_1 b_{NN}^{(1)} \\ a_{NN}^{(2)} & \lambda_2 b_{NN}^{(2)} \end{bmatrix}\right) = 0, \tag{4.43}$$

which yields

$$\lambda_2 = \lambda_1 \frac{a_{NN}^{(2)} b_{NN}^{(1)}}{a_{NN}^{(1)} b_{NN}^{(2)}}. \tag{4.44}$$

Eliminating the eigenvalue λ_2 from the system, we may now solve (4.28) by using the knowledge of the total mass of the system, ie.

$$M = p_1\beta_1 + p_2\beta_2 + \ldots + p_N\beta_N, \tag{4.45}$$

where p_i are constant factors given by the actual discrete model. Following the reconstruction procedure from the previous chapter we replace one of the rows in (4.28) by (4.45) and in this way transform (4.28) to the non-homogeneous system

$$\begin{bmatrix} \mathbf{A}^{(1)} & -\lambda_1\mathbf{B}^{(1)} \\ \mathbf{C}^{(1)} & \mathbf{C}^{(2)} \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \mathbf{m} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{d} \end{Bmatrix}, \tag{4.46}$$

where

$$\mathbf{C}^{(1)} = \begin{cases} \mathbf{A}^{(2)}, & \text{for the first } (n-1) \text{ rows} \\ \mathbf{0}, & \text{for the last row} \end{cases}, \tag{4.48}$$

$$\mathbf{C}^{(2)} = \begin{cases} -\lambda_2\mathbf{B}^{(2)}, & \text{for the first } (n-1) \text{ rows} \\ \mathbf{p}, & \text{for the last row} \end{cases}, \tag{4.49}$$

$$\mathbf{p} = \{p_1 \; p_2 \ldots p_N\}^T.$$

and

$$\mathbf{d} = \begin{cases} \mathbf{0}, & \text{for the first } (n-1) \text{ elements} \\ M, & \text{for the last element} \end{cases}. \tag{4.50}$$

The system (4.47) is non-homogeneous and may be therefore use to evaluate unknown parameters $\alpha_1, \alpha_2, ..., \alpha_N, \beta_1, \beta_2, ..., \beta_N$. This may be done simply by finding the inverse of the coefficient matrix or using some of the numerical methods.

The above procedure is summarised in the following algorithm.

Algorithm 4.1

Suppose that for a system of n degrees of freedom, an eigenvalue λ_1, two eigenvectors Φ_1, Φ_2 and the total mass M are given. We reconstruct the unknown physical parameters of the model by the following

a) Set n, λ_1, Φ_1, Φ_2 and M.

b) Define the matrices $\mathbf{G}_{(e)}$ and $\mathbf{Y}_{(e)}$.

c) Set the matrices $\mathbf{A}_{(e)}$ and $\mathbf{E}$ and evaluate $\mathbf{P}_{(e)}$ and $\mathbf{Q}_{(e)}$ from (4.8) and (4.9).

d) Using (4.18) - (4.21) evaluate $\mathbf{a}_{(e)}^{(1)}$, $\mathbf{b}_{(e)}^{(1)}$, $\mathbf{a}_{(e)}^{(2)}$ and $\mathbf{b}_{(e)}^{(2)}$.

e) Form the matrices $\mathbf{A}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{B}^{(2)}$, as defined in (4.24).

f) Find the eigenvalue λ_2 from the condition (4.40).

g) Determine the matrices $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ and the vector $\mathbf{d}$ by (4.48) - (4.50).

h) Solve the system (4.46) and evaluate the unknown parameters α_i and β_i, $i=1, 2, \ldots, n$.

In the next section the above solution is applied to some particular cases.

4.3.1 The Finite Difference model of a vibrating rod

The eigenvalue problem corresponding to the finite difference model of the rod is given by

$$\mathbf{Ku} = \lambda\mathbf{Mu}\,, \tag{4.51}$$

The stiffness matrix **K** and the mass matrix **M** of a n degrees-of-freedom system in a fixed-free configuration are given by

$$\mathbf{K} = \begin{bmatrix} (k_1 + k_2) & (-k_2) & & & \\ (-k_2) & (k_2 + k_3) & (-k_3) & & \\ & \ddots & & & \\ & & & (k_{n-1} + k_n) & (-k_n) \\ & & & (-k_n) & (k_n) \end{bmatrix} \tag{4.52}$$

and

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{n-1} & \\ & & & & m_n \end{bmatrix}, \tag{4.53}$$

where

$$k_i = \frac{(EA)_i}{h} \text{ and } m_i = (\rho A)_i h \,.\, i=1,2,\ldots,n \tag{4.54}$$

To obtain the above matrices from (4.6) and (4.7) of the general model we introduce the matrices $\mathbf{A}_i$ as

$$\mathbf{A}_i = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}, \; a_{mn} = \begin{cases} 1, & m = 1, n = i \\ 1, & m = 2, n = i+1 \\ 0, & \textit{elsewhere} \end{cases}. \tag{4.55}$$

Using (4.54) we therefore have

$$\mathbf{K} = \sum_{e=1}^{n} \frac{(EA)_{(e)}}{h} \mathbf{A}_e^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{A}_e \tag{4.55}$$

and

$$\mathbf{M} = \sum_{e=1}^{n} (\rho A)_{(e)} h \mathbf{A}_e^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{A}_e \,. \tag{4.56}$$

For the fixed-free boundary conditions, the first row and column of (4.55) and (4.56) must be removed, so the last two expressions yields (4.52) and (4.53). The unknown physical parameters are thus in this case given by

$$\alpha_{(e)} = \frac{(EA)_{(e)}}{h} \text{ and } \beta_{(e)} = (\rho A)_{(e)} h, \tag{4.57a}$$

and

$$\mathbf{K}_{(e)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{and} \quad \mathbf{M}_{(e)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.57b}$$

In order to reconstruct these parameters, the algorithm for the general case may be applied using (4.55)-(4.56) without further modifications.

4.3.2 The Finite Element model of a vibrating rod

For a finite element model of the rod, the element stiffness and mass matrices are given by

$$\mathbf{A}_{(e)} = \frac{(EA)_{(e)}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.58}$$

and

$$\mathbf{B}_{(e)} = \frac{h(\rho A)_{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4.59}$$

Following the solution to the problem for the general case, this yields

$$\alpha_{(e)} = \frac{(EA)_{(e)}}{h}, \ \beta_{(e)} = \frac{(\rho A)_{(e)} h}{6}, \tag{4.60a}$$

$$\mathbf{K}_{(e)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{and} \quad \mathbf{M}_{(e)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{4.60b}$$

The global stiffness and mass matrices are therefore

$$\mathbf{K} = \sum_{e=1}^{n} \frac{(EA)_{(e)}}{h} \mathbf{A}_e^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{A}_e \tag{4.61}$$

and

$$\mathbf{M} = \sum_{e=1}^{n} \frac{h(\rho A)_{(e)}}{6} \mathbf{A}_e^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{A}_e \,, \tag{4.62}$$

where the matrices $\mathbf{A}_e$ are of the same form as in the case of the finite difference model.

4.3.3 The finite difference model of a clamped-free vibrating beam

A finite difference model, or equivalently, a discrete mass-spring-rod model of a clamped-free vibrating beam is given by

$$\mathbf{E}\mathbf{H}^{-1}\mathbf{E}\mathbf{K}_1\mathbf{E}^T\mathbf{H}^{-1}\mathbf{E}^T\mathbf{u} = \lambda\mathbf{M}\mathbf{u}\,, \tag{4.63}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}_{(NxN)}, \tag{4.64}$$

$$h_i = x_i - x_{i-1}, \quad k_i = \frac{r(x_i)}{h_i}, \quad m_i = \rho(x_i)h_i$$

$$\mathbf{H}=\mathrm{diag}(h_i),\ \mathbf{K}_1=\mathrm{diag}(k_i),\ \mathbf{M}=\mathrm{diag}(m_i),\ i=1, 2, \ldots, N.$$

If the elements are of the same length, h_i=const. $i=1,2,\ldots,N$, this system may be written as

$$\mathbf{Ku} = \lambda\mathbf{Mu}\,, \tag{4.65}$$

where

$$\mathbf{K} = \mathbf{E}\mathbf{E}\,\mathbf{K}_1\,\mathbf{E}^T\mathbf{E}^T \tag{4.66}$$

Introducing the matrices

$$\mathbf{A}_i = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix},\ a_{mn} = \begin{cases} 1,\ m=1, n=i \\ 1,\ m=2, n=i+1, \\ 0,\ \textit{elsewhere} \end{cases} \tag{4.67}$$

the stiffness and mass matrices may be written in the form

$$\mathbf{K} = \mathbf{E}\mathbf{E}\sum_{e=1}^{N}\frac{k_{(e)}}{h^2}\mathbf{A}_e^T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A}_e\,\mathbf{E}^T\mathbf{E}^T. \tag{4.68}$$

and

$$\mathbf{M} = \sum_{e=1}^{N}(\rho A)_{(e)}h\mathbf{A}_e^T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A}_e\,. \tag{4.69}$$

The unknown physical parameters are thus given by

$$\alpha_{(e)} = \frac{k_{(e)}}{h^2} \text{ and } \beta_{(e)} = \rho_{(e)} h. \tag{4.59}$$

4.4 Numerical Examples

In this section we apply the proposed solution for the general case on some typical models, ie. the finite element and the finite difference models of rod and beam.

4.4.1 The finite element model of a uniform rod with given data corresponding to the continuous model

In this example we solve the inverse vibration problem for a finite element model of a uniform rod, fixed at one end and free to vibrate at the other. In order to have a realistic situation, we take the eigendata that corresponds to the continuous model of the system. The eigenvalue problem for the continuous model of the rod is given by

$$\begin{cases} \dfrac{d}{dx}\left(E(x)A(x)\dfrac{du(x)}{dx} \right) + \lambda\rho(x)A(x)u(x) = 0, \ 0 < x < L \\ \\ u(x)\Big|_{x=0} = 0 \text{ and } \dfrac{du(x)}{dx}\Big|_{x=L} = 0 \end{cases} \tag{4.63}$$

In the case where the rod is uniform, say $k(x)=m(x)=1$, this equation may be solved analytically. The solution for the eigenvalues and their corresponding eigenfunctions are given by

$$\lambda_i = \left(\frac{(2i-1)\pi}{2}\right)^2, \quad u^{(i)}(x) = \sin\left(\frac{(2i-1)\pi}{2}x\right), \tag{4.64}$$

where $i=1, 2, \ldots$. In the discrete form, the eigenfunctions $u^{(i)}(x)$ may be written as

$$u^{(i)} = \sin\left(\frac{(2i-1)\pi}{2}x_i\right), \tag{4.65}$$

where x_i correspond to the nodal points of the model. For model order $N=5$ this gives

$$\lambda_1=2.4674, \; u^{(1)} = \sin\left(\frac{\pi}{2}x_i\right) \text{ and } u^{(2)} = \sin\left(\frac{3\pi}{2}x_i\right),$$

where $x_i = \left\{\frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots\right\}$. In the following we use the above eigendata to solve the inverse problem for a discrete rod. Suppose that

$$\lambda_1 = 2.4674,$$

$$\mathbf{u}_1^T = \{0.3090 \;\; 0.5878 \;\; 0.8090 \;\; 0.9511 \;\; 1.0000\},$$

$$\mathbf{u}_2^T = \{0.8090 \;\; 0.9511 \;\; 0.3090 \;\; -0.5878 \;\; -1.0000\} \text{ and}$$

$$M = 5$$

are given, where M is the total mass. We first evaluate the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$, ie.

$$\mathbf{A}_1 = \begin{bmatrix} 0.3090 & -0.2788 & 0 & 0 & 0 \\ 0 & 0.2788 & -0.2212 & 0 & 0 \\ 0 & 0 & 0.2212 & -0.1420 & 0 \\ 0 & 0 & 0 & 0.1420 & -0.0489 \\ 0 & 0 & 0 & 0 & 0.0489 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} -1.5249 & -2.9752 & 0 & 0 & 0 \\ 0 & -3.6631 & -4.8968 & 0 & 0 \\ 0 & 0 & -5.4426 & -6.3390 & 0 \\ 0 & 0 & 0 & -6.6894 & -7.1607 \\ 0 & 0 & 0 & 0 & -7.2814 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.8090 & -0.1420 & 0 & 0 & 0 \\ 0 & 0.1420 & 0.6420 & 0 & 0 \\ 0 & 0 & -0.6420 & 0.8968 & 0 \\ 0 & 0 & 0 & -0.8968 & 0.4122 \\ 0 & 0 & 0 & 0 & -0.4122 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} -1.6180 & -2.5691 & 0 & 0 & 0 \\ 0 & -2.7111 & -2.2111 & 0 & 0 \\ 0 & 0 & -1.5691 & -0.0302 & 0 \\ 0 & 0 & 0 & -0.8666 & 2.1756 \\ 0 & 0 & 0 & 0 & 2.5878 \end{bmatrix}.$$

Then we determine the eigenvalue λ_2 from (4.40), getting $\lambda_2 = 23.6983$, and by reconstruct the unknown physical parameters by solving the system (4.46). The results are given below

NODE	$(EA)_{(e)}$	$(\rho A)_{(e)}$	EXACT
1	0.9918	1.0000	1.0000
2	0.9918	1.0000	1.0000
3	0.9918	1.0000	1.0000
4	0.9918	1.0000	1.0000
5	0.9918	1.0000	1.0000

Table 4-1. The reconstructed physical parameters

The exact values for k_i and m_i corresponding to the continuous model of the rod are $k_i = m_i = 1.0000$. We see that uniform rod parameters have been identified, with small errors of about 0.8 % for the stiffnesses. This is because the finite difference model for a uniform rod gives eigenvectors which coincide with the eigenfunction of the continuous system. In general, however, for a non-uniform rod less favourite results may be expected. We now consider the inverse problem for a non-uniform rod.

4.4.2 A non-uniform fixed-free rod with $A(x) = e^x$

Consider an axially vibrating rod in a fixed-free configuration and let the cross-sectional area of the rod be given by:

$$A(x) = e^x. \tag{4.66}$$

Suppose that the Young's modulus of elasticity and the mass per unit length are constants. With the above assumptions, the eigenvalue problem (4.63) may be solved analytically. Substituting the expression for $A(x)$ in (4.63), we find

$$\frac{d}{dx}\left(Ee^{x}\frac{du(x)}{dx}\right)+\lambda\rho e^{x}u(x)=0, \tag{4.67}$$

Evaluating the above equation and taking into account the boundary conditions, yields

$$\begin{cases} u''+u'+\lambda\dfrac{\rho}{E}u=0 \\ u(0)=0 \\ u'(1)=0 \end{cases}, \tag{4.68}$$

where primes denote derivatives with respect to x. To solve the above problem we transform (4.67) to the standard form

$$\ddot{\xi}+2\delta\omega_n\dot{\xi}+\omega_n^2\xi=0, \tag{4.69}$$

by setting $\xi=u$, $2\delta\omega_n = 1$ and $\omega_n^2=\lambda\dfrac{\rho}{E}$. The above equation has a solution given by

$$\xi(t)=e^{-\delta\omega_n t}(A\sin\omega_n\sqrt{1-\delta^2}t+B\cos\omega_n\sqrt{1-\delta^2}t),\quad \xi<1. \tag{4.70}$$

Taking into account that $\omega_n\sqrt{1-\delta^2}\,t = \sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}$, we obtain

$$u(x) = e^{-\frac{1}{2}x}\left(A\sin\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}x + B\cos\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}x\right). \tag{4.71}$$

The boundary condition at the fixed end, $u(0) = 0$, gives $B = 0$. At the free end the condition $u'(1) = 0$ gives

$$A\left(-\frac{1}{2}e^{-\frac{1}{2}}\sin\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}} + \sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}e^{-\frac{1}{2}}\cos\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}\right) = 0. \tag{4.73}$$

For a non-trivial solution $u(x)\neq 0$, the constant A must not be equal to zero. This yields

$$\tan\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}} = 2\sqrt{\lambda\frac{\rho}{E}-\frac{1}{4}}. \tag{4.74}$$

Equation (4.74) is the frequency equation for the eigenvalue problem (4.68). Its roots can be determined graphically by drawing the function $y(\eta) = \tan\sqrt{\eta-\frac{1}{4}}$ and finding its intersections with $g(\eta) = 2\sqrt{\eta-\frac{1}{4}}$ using the Bisection method. Applying this method we found

$$\lambda_1 = 1.6085\frac{E}{\rho}, \tag{4.75a}$$

$$\lambda_2 = 21.4488\frac{E}{\rho}, \tag{4.75b}$$

and

$$\lambda_3 = 60.9323\frac{E}{\rho} \tag{4.75c}$$

The eigenfunction of the system are

$$u_i(x) = A_i e^{-\frac{1}{2}x} \sin\sqrt{\lambda_i \frac{\rho}{E} - \frac{1}{4}}x, \quad i = 1,2,\ldots. \tag{4.76}$$

We now solve the problem of reconstructing the physical parameters of the rod, using the finite difference and the finite element models.

4.4.2.1 A finite difference model for the exponential rod

Let us divide the system into $n=5$ finite differences of the same length $h=1/5$, as shown in the *Figure* 4-1. We choose the nodal points $i=1, 2, 3, 4, 5$ corresponding to the midpoint of each finite difference segment. We define the element stiffness and mass parameters of the model as

$$k_i = \frac{E_{i-1/2}e^{(i-1/2)h}}{h} \quad \text{and} \quad m_i = \rho_i e^{ih} h \tag{4.77}$$

Suppose that one eigenvalue, two eigenvectors and the total mass corresponding to the *continuous* model of the rod are given. Using (4.77) and $n=5$, $L=1$ and $h=0.2$ this gives

$\lambda_1 = 1.6085,$

$\mathbf{u}_1^T = \{\ 0.1106 \quad 0.2949 \quad 0.4286 \quad 0.5132 \quad 0.5528\},$

$\mathbf{u}_2^T = \{\ 0.4227 \quad 0.8453 \quad 0.5797 \quad -0.0573 \quad -0.5373\},$

$M = 8.58.$

To solve the problem we apply the algorithm 4-1 and evaluate the following matrices $\mathbf{A}_i$, $\mathbf{B}_i$

$$\mathbf{A}_1 = \begin{bmatrix} 0.2212 & -0.1842 & 0 & 0 & 0 \\ 0 & 0.1842 & -0.1337 & 0 & 0 \\ 0 & 0 & 0.1337 & -0.0846 & 0 \\ 0 & 0 & 0 & 0.0846 & -0.0359 \\ 0 & 0 & 0 & 0 & 0.0359 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} -0.1779 & 0 & 0 & 0 & 0 \\ 0 & -0.4743 & 0 & 0 & 0 \\ 0 & 0 & -0.6894 & 0 & 0 \\ 0 & 0 & 0 & -0.8256 & 0 \\ 0 & 0 & 0 & 0 & -0.8891 \end{bmatrix},$$

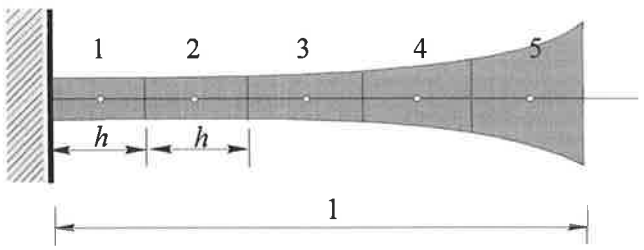

Figure 4-1 The finite difference model of the rod with $A=e^x$

$$\mathbf{A}_2 = \begin{bmatrix} 0.8453 & -0.4226 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.2656 & 0 & 0 \\ 0 & 0 & -0.2656 & 0.6369 & 0 \\ 0 & 0 & 0 & -0.6369 & 0.4800 \\ 0 & 0 & 0 & 0 & -0.4800 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} -0.4227 & 0 & 0 & 0 & 0 \\ 0 & -0.8456 & 0 & 0 & 0 \\ 0 & 0 & -0.5797 & 0 & 0 \\ 0 & 0 & 0 & -0.0573 & 0 \\ 0 & 0 & 0 & 0 & 0.5373 \end{bmatrix}.$$

The second eigenvalue is found to be

$$\lambda_2 = 20.0966.$$

Then we solved the system (4.46) and obtained the following values for the physical parameters

NODE	$(EA)_{(e)}$	$(\rho A)_{(e)}$	EXACT
1	1.0540	1.1008	1.1052
2	1.2231	1.3448	1.3499
3	1.4941	1.6435	1.6487
4	1.8255	2.0104	2.0138
5	2.2293	2.4776	2.4596

Table 4-2. The reconstructed physical parameters

The maximum relative error for the stiffnesses is 9.4% and for the masses 0.4%. A more accurate solution may be obtained if the number of the segments n are increased. For example, for the model with $n = 10$ elements, suppose that

$$\mathbf{u}_1^T = \{0.0568\ \ 0.1614\ \ 0.2535\ \ 0.3330\ \ 0.3999\ \ 0.4543\ 0.4965\ 0.5272\ 0.5468\ 0.5563\},$$

$$\mathbf{u}_2^T = \{0.2225\ 0.5910\ 0.8059\ 0.8388\ \ 0.7003\ \ 0.4347\ \ 0.1072\ -0.2107\ -0.4560\ -0.586\},$$

$$M = 8.58$$

are given. Using this data in the algorithm 4-1, we obtained $\lambda_2 = 21.0982$. The values for the physical parameters are then found to be

NODE	$(EA)_{(e)}$	$(\rho A)_{(e)}$	EXACT
1	0.5129	0.5252	0.5256
2	0.5527	0.5805	0.5809
3	0.6108	0.6415	0.642
4	0.6751	0.7090	0.7095
5	0.7461	0.7836	0.7841
6	0.8246	0.8660	0.8666
7	0.9113	0.9571	0.9577
8	1.0072	1.0579	1.0585
9	1.1130	1.1694	1.1698
10	1.2298	1.2974	1.2928

Table 4-3. The reconstructed physical parameters

The maximum relative error for the stiffnesses is now 4.9% and for the masses 0.08%. As expected, there is a discrepancy between the exact and the identified model in the non-uniform case. This discrepancy is acceptable when considering engineering applications.

4.4.2.2 The finite element model

The finite element model of the system is similar to the finite difference model from the previous example, but now we set the nodal points at the end of each element, as shown in *Figure* 4-2. Suppose that for $L=1$, $n=5$ and $h=0.2$ one eigenvalue, two eigenvectors and the total mass corresponding to the continuous model of the rod are given by

$$\lambda_1 = 1.6085,$$

$$\mathbf{u}_1 = \{\, 0.2090 \quad 0.3680 \quad 0.4769 \quad 0.5383 \quad 0.5574 \}^T,$$

$$\mathbf{u}_2 = \{0.7319 \quad 0.7787 \quad 0.2293 \quad -0.3936 \quad -0.6065\}^T \text{ and } M = 9.48.$$

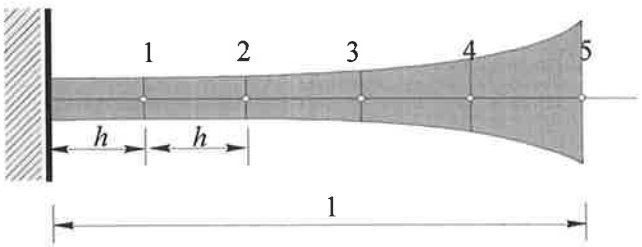

Figure 4-2 The finite elemet model of the rod with $A=e^x$

Algorithm 4.1 produces the following matrices

$$\mathbf{A}_1 = \begin{bmatrix} 0.2090 & -0.1590 & 0 & 0 & 0 \\ 0 & 0.1590 & -0.1088 & 0 & 0 \\ 0 & 0 & 0.1088 & -0.0614 & 0 \\ 0 & 0 & 0 & 0.0614 & -0.0191 \\ 0 & 0 & 0 & 0 & 0.0191 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} -0.6724 & -1.2644 & 0 & 0 & 0 \\ 0 & -1.5202 & -1.9510 & 0 & 0 \\ 0 & 0 & -2.1261 & -2.4000 & 0 \\ 0 & 0 & 0 & -2.4988 & -2.6284 \\ 0 & 0 & 0 & 0 & -2.6591 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.7204 & -0.0685 & 0 & 0 & 0 \\ 0 & 0.0685 & 0.5147 & 0 & 0 \\ 0 & 0 & -0.5147 & 0.6198 & 0 \\ 0 & 0 & 0 & -0.6198 & 0.2573 \\ 0 & 0 & 0 & 0 & -0.2573 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} -1.4407 & -2.2296 & 0 & 0 & 0 \\ 0 & -2.2981 & -1.8519 & 0 & 0 \\ 0 & 0 & -1.3372 & -0.2026 & 0 \\ 0 & 0 & 0 & 0.4172 & 1.2943 \\ 0 & 0 & 0 & 0 & 1.5516 \end{bmatrix}.$$

The second eigenvalue is determined by (4.40), which gives $\lambda_1 = 23.1040$. Finally, by solving equation (4.46), we determine the following physical parameters

NODE	$(EA)_{(e)}$	$(\rho A)_{(e)}$	EXACT
1	1.2092	1.1888	1.2214
2	1.4779	1.4733	1.4918
3	1.8082	1.7895	1.8221
4	2.2088	2.2318	2.2255
5	2.5967	2.7957	2.7183

Table 4-4. The reconstructed physical parameters

Again, we may improve the accuracy by increasing the number of finite elements. For the model with $n=10$ elements, suppose that the given eigendata consists of

$\lambda_1 = 1.6085,$

$\mathbf{u}_1 = \{0.1106\ 0.2090\ 0.2949\ 0.3680\ 0.4286\ \ 0.4769\ \ 0.5132\ 0.5383\ \ 0.5528\ \ 0.5574\}^T,$

$\mathbf{u}_2 = \{0.4227\ 0.7204\ \ 0.8453\ 0.7889\ \ .5797\ 0.2741\ -0.0573\ -0.3457\ -0.5373\ -0.6030\}^T$

$M = 9.03.$

For this case the algorithm gives $\lambda_2 = 21.8247$, and produces the following values for the physical parameters

NODE	$(EA)_{(e)}$	$(\rho A)_{(e)}$	EXACT
1	0.5509	0.5513	0.5526
2	0.6088	0.6082	0.6107
3	0.6729	0.6729	0.6749
4	0.7437	0.7431	0.7459
5	0.8220	0.8221	0.8243
6	0.9085	0.9080	0.9110
7	1.0041	1.0050	1.0069
8	1.1097	1.1101	1.1127
9	1.2246	1.2332	1.2298
10	1.3265	1.3741	1.3591

Table 4-5. The reconstructed physical parameters

4.4.3 The finite difference model of a beam

Consider a beam fixed at $x=0$ and free to oscillate at $x=1$. For $n=5$ and $h=1/5$, let the following data be given

$$\lambda_1 = 10,$$

$$\mathbf{u}^{(1)} = \{\, 0.08 \;\; 0.20 \;\; 0.36 \;\; 0.54 \;\; 0.73 \,\}^T,$$

$$\mathbf{u}^{(2)} = \{\, 0.35 \;\; 0.60 \;\; 0.45 \;\; 0.05 \;\; -0.50 \,\}^T \text{ and } M=1.$$

The algorithm produces the following matrices

$$\mathbf{A}_1 = \begin{bmatrix} 0.0800 & -0.0800 & 0 & 0 & 0 \\ 0 & 0.0400 & -0.0800 & 0.0200 & 0 \\ 0 & 0 & 0.0400 & -0.0400 & 0.0100 \\ 0 & 0 & 0 & 0.0200 & -0.0200 \\ 0 & 0 & 0 & 0 & 0.0100 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} -0.8000 & 0 & 0 & 0 & 0 \\ 0 & -2.0000 & 0 & 0 & 0 \\ 0 & 0 & -3.6000 & 0 & 0 \\ 0 & 0 & 0 & -5.4000 & 0 \\ 0 & 0 & 0 & 0 & -7.3000 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.3500 & 0.2000 & 0 & 0 & 0 \\ 0 & -0.1000 & 0.8000 & -0.2500 & 0 \\ 0 & 0 & -0.4000 & 0.5000 & -0.1500 \\ 0 & 0 & 0 & -0..2500 & 0.3000 \\ 0 & 0 & 0 & 0 & -0.1500 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} -0.3500 & 0 & 0 & 0 & 0 \\ 0 & -0.6000 & 0 & 0 & 0 \\ 0 & 0 & -0.4500 & 0 & 0 \\ 0 & 0 & 0 & -0.0500 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 \end{bmatrix}.$$

The second eigenvalue is determined from the condition (4.40). This gives $\lambda_2=219$ and the algorithm evaluate the following values for the physical parameters $k_i=r_i/h$ and m_ih

NODE	k_i	m_i
1	4.8124	0.4089
2	6.9165	0.2972
3	4.5352	0.0586
4	5.4968	0.0747
5	4.6898	0.1604

Table 4-6. The reconstructed physical parameters

Using these values we reconstruct the stiffness and mass matrices of the system

$$\mathbf{K} = \begin{bmatrix} 925.3374 & -572.5833 & 113.3799 & 0 & 0 \\ -572.5833 & 763.8514 & -501.6000 & 137.4202 & 0 \\ 113.3799 & -501.6000 & 780.3048 & -509.3289 & 117.2443 \\ 0 & 137.4202 & -509.3289 & 606.3973 & -234.4886 \\ 0 & 0 & 117.2443 & -234.4886 & 117.2443 \end{bmatrix}$$

and

$$\mathbf{M} = \begin{bmatrix} 0.4089 & & & & \\ & 0.2972 & & & \\ & & 0.0586 & & \\ & & & 0.0747 & \\ & & & & 0.1606 \end{bmatrix}.$$

In order to verify these results, the eigenvalue problem (4.3) has been solved using the above matrices. The resulting five eigenvalues and their corresponding eigenvectors are given in *Table* 4-7.

We note that the first eigenvector $\phi^{(1)} = \{ 0.08 \;\; 0.20 \;\; 0.36 \;\; 0.54 \;\; 0.73 \}^T$ is identical to the eigenvector $\mathbf{u}^{(1)}$. Upon the normalisation $\phi^{(2)} = \phi^{(2)} 0.3500 / 0.3615$ we obtain as required $\phi^{(2)} = \{ 0.35 \;\; 0.60 \;\; 0.45 \;\; 0.05 \;\; -0.50 \}^T$.

NODE	μ_1=10.0	μ_2=219.0	μ_3=2182.9	μ_4=4672.4	μ_5=19918.0
	$\phi^{(1)}$	$\phi^{(2)}$	$\phi^{(3)}$	$\phi^{(4)}$	$\phi^{(5)}$
1	0.0800	0.3615	0.4089	0.2666	0.0211
2	0.2000	0.6197	-0.1051	-0.3828	-0.0983
3	0.3599	0.4648	-0.6494	0.3830	0.8440
4	0.5399	0.0516	-0.5786	0.7682	-0.5218
5	0.7298	-0.5164	0.2552	-0.2136	0.0718

Table 4-7. The eigenvalues and their corresponding eigenvectors

4.4.4 A structural dynamic modification for a high building

The study of the dynamics of structures leads to difficulties associated with the discrepancy between the values of the modal parameters obtained by the analytical model (ie. finite element model) and the actual measurements from the prototype. There is a significant amount of literature on this subject (see eg. [6], [8], [9], [50]) where several methods of attempting to correct the model are suggested. The correction is usually done by defining a new model which is as close as possible to the original model and which closely matches the experimental data. The method

developed in this chapter introduces a new significant improvement in the above mentioned methods as we take the *connectivity* of the system into account. ie. the actual non-zero elements in the mass and stiffness matrices are conserved.

To demonstrate the actual improvement obtained by the method presented here we use experimental data obtained by Lihua and Shangwen [50]. This data is shown in *Table* 4-8. Algorithm 4.1 has been applied with this data as input and an analytical model has been determined.

NODE	λ_1=0.6348	λ_2=1.5652	λ_3=2.7100
	$u^{(1)}$	$u^{(2)}$	$u^{(4)}$
1	0.098	-0.237	0.301
2	0.219	-0.516	0.571
3	0.351	-0.706	0.505
4	0.474	-0.801	0.220
5	0.616	-0.598	-0.464
6	0.736	-0.267	-0.843
7	0.835	0.123	-0.727
8	0.927	0.571	0.115
9	0.977	0.858	0.580
10	1.000	1.000	1.000

Table 4-8. Experimental data of vibrating structure taken from [50]

A comparison between the results obtained in [50], the method presented here, and the real physical data, shown in *Table* 4-9, demonstrates a clear improvement in the model identification when our method is used. The maximum relative error obtained by [50] is 22.7% (average 5.9%), whereas with our method the maximum relative error is only 2.6% (average 1.3%).

In addition to the benefit of conserving the physical connectivity in our model we can estimate both stiffnesses and masses. The method suggested by Lihua and Shangwen [50] can estimate stiffnesses only. In summary, the above example suggests that a significant improvement can be made when our method is used to correct an analytical model using experimental data.

NODE	STIFFNESS			MASS		
	BY REF. [50]	BY PRESENT	REAL	BY REF. [50]	BY PRESENT	REAL
1	912.6	947.0	950	-	0.9	1.0
2	717.4	755.0	750	-	1.5	1.5
3	643.6	652.8	650	-	1.0	1.0
4	632.2	656.1	650	-	2.0	2.0
5	474.8	461.4	450	-	1.0	1.0
6	510.7	460.5	450	-	0.9	0.9
7	434.6	448.8	440	-	0.8	0.8
8	362.5	365.6	360	-	1.1	1.0
9	382.1	371.1	370	-	0.7	0.7
10	262.8	332.6	340	-	0.5	0.5

Table 4-10. The reconstructed physical parameters

4.5 Summary

In this chapter a method for the solution of the inverse problem for general discrete models of vibrating systems is developed. This may be a finite difference, a finite element, or a lumped-mass model of one-dimensional systems such as mass-spring systems, rods or beams. It has been shown that the physical parameters of the system may be reconstructed from the knowledge of one eigenvalue, two eigenvectors and the total mass of the system. Since the actual form of **K** and **M** is not needed to be specified, the reconstruction algorithm is applicable as a general solution. This general solution is then applied to some particular vibrating systems. It is demonstrated by means of numerical examples that the solution may be successfully applied for the case of finite difference and finite element models of non-uniform rods and beams, and for discrete models of multi-storey buildings.

5 Inverse problem for some two and three dimensional vibratory systems

5.1 Introduction

In the previous two chapters, the inverse problem is solved for a higher order finite difference model of a rod and for the case of a general discrete model of one-dimensional vibrating systems. In these systems, we assume that the masses oscillate in line, ie. in a single direction of motion. As a consequence, the nodal points that are used to represent the motion of these systems may be naturally ordered by their relative distance from one end, and the natural frequencies are distinct.

This chapter is concerned with the inverse vibration problem for some multi-dimensional systems. Obviously, these systems are more complex and difficult to analyse. Analytical solutions for a direct vibrating problem may be obtained only for some particular systems of simple geometry. One such system is the model of a stretched membrane. In the following, we first derive the differential equation describing the small vibration of the membrane, and show how to obtain a finite difference approximation of the eigenvalue problem. Then we pose the inverse problem for this model and develop a reconstruction procedure using one

eigenvalue, three eigenvectors and the total mass. Following this, we consider a simply connected three-dimensional vibrating lattice. The problem of reconstructing the physical parameters from the knowledge of one eigenvalue, four eigenvectors and the total mass of the model is solved. Two numerical examples are presented.

5.2 The differential equation for a membrane

We suppose that the membrane is stretched and that it is fixed along its boundaries in the x-y plane. If the membrane is non-uniform, the tension per unit length T may vary from point to point of the system, eg. $T = T(x,y)$. In addition, we suppose that the deflection $w(x,y,t)$ in the z direction is small compared to the dimensions of the membrane, so that all angles of inclination are small. The deflections in other directions are neglected (*Figure* 5-1) and thus the vibration of the system is linear.

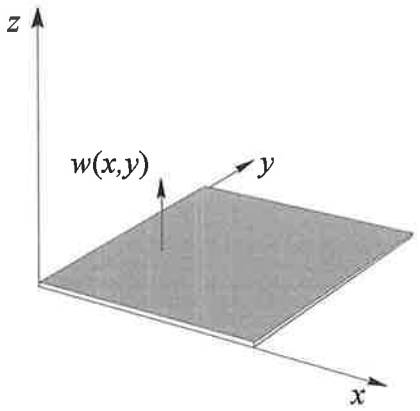

Figure 5-1 Transverse vibrations of a thin stretched membrane

Figure 5-2 shows an infinitely small portion of the membrane and the forces acting on its edges. In the z direction, the tension Q acts on the element with the angles α and β, as shown in *Figure* 5-3. With the assumption of small vibrations about the equilibrium, the horizontal components of the forces on the edges of the element may be neglected. The vertical components are $-Qdy\sin\alpha$ and $\left(Q+\frac{\partial Q}{\partial x}dx\right)dy\sin\beta$. Since the angles α and β are small, we use the approximation sin α ≅ tan α and similar expression for β. The resulting force of the vertical component is then

$$-Qdy\sin\alpha+\left(Q+\frac{\partial Q}{\partial x}dx\right)dy\sin\beta=-Qdy\tan\alpha+\left(Q+\frac{\partial Q}{\partial x}dx\right)dy\tan\beta$$

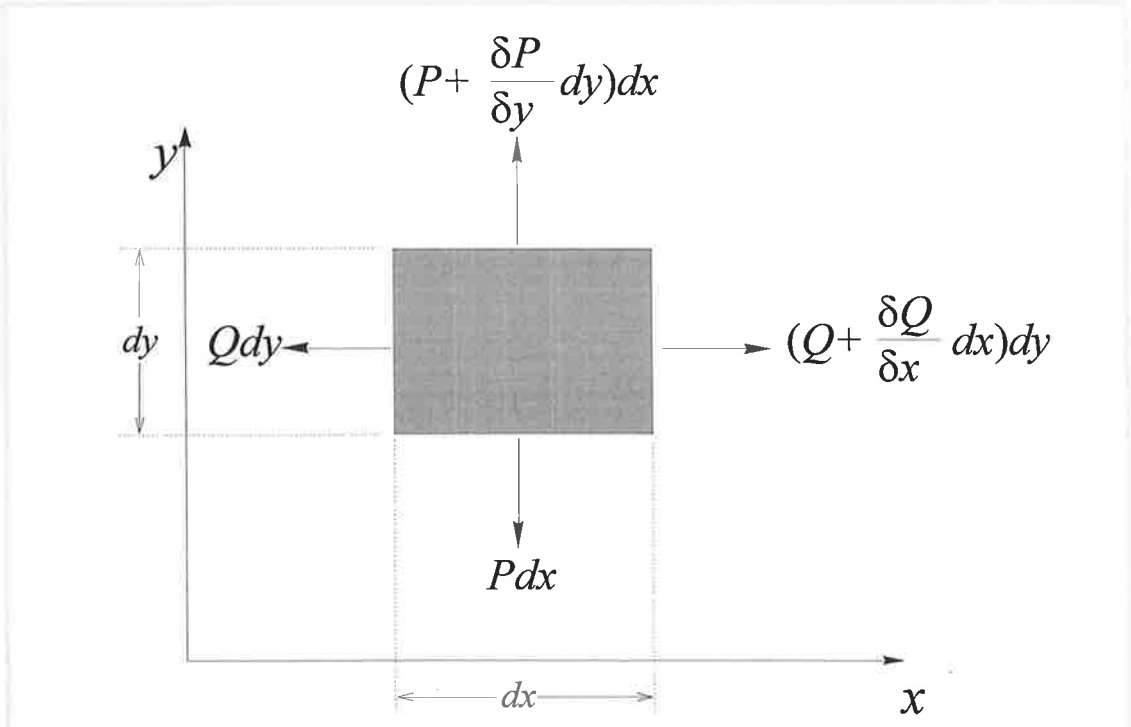

Figure 5-2 Acting forces on a small portion of the membrane in the *x-y* plane

$$= -Qdy\, w_x + \left(Q + \frac{\partial Q}{\partial x} dx\right) dy \left(w_x + \frac{\partial w_x}{\partial x} dx\right), \tag{5.1}$$

where w_x denotes the derivatives of w with respect to x.

Similarly, the vertical components of the forces acting on the other two edges are - $P\, dx \sin\gamma$ and $\left(P + \frac{\partial P}{\partial y} dy\right) dx \sin\delta$. The resulting force of these two components may thus be approximated similarly as

$$-Pdx\, w_y + \left(P + \frac{\partial P}{\partial y} dy\right) dx \left(w_y + \frac{\partial w_y}{\partial y} dy\right). \tag{5.2}$$

The Newton's second law yields

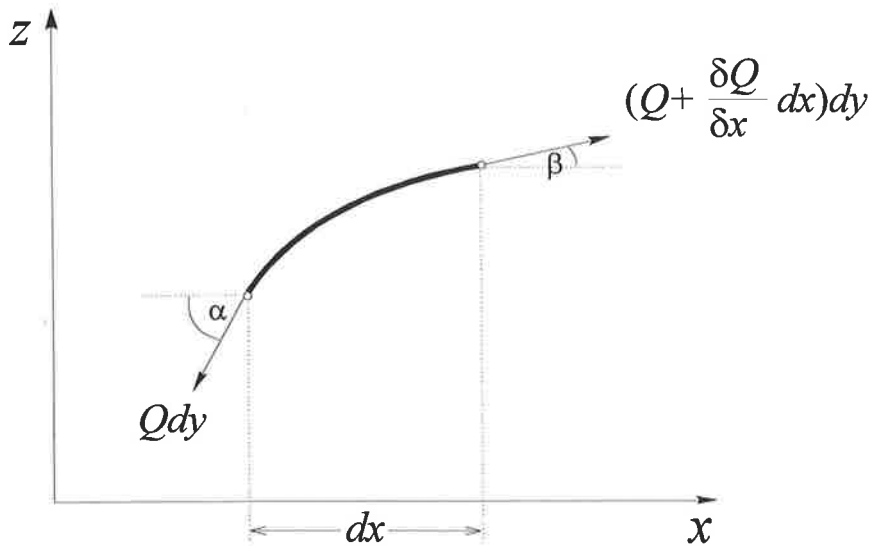

Figure 5-3 Acting forces in the *z-x* plane

$$\sum F_z = ma_z = \rho(x,y)dxdy\frac{\partial^2 w}{\partial x^2}, \tag{5.3}$$

where $\rho(x,y)$ is the mass per unit area of the membrane. Substituting (5.1) and (5.2) into the above equation, we get

$$-Qdy\, w_x + \left(Q + \frac{\partial Q}{\partial x}dx\right)dy\left(w_x + \frac{\partial w_x}{\partial x}dx\right) - Pdx\, w_y + \left(P + \frac{\partial P}{\partial y}dy\right)dx\left(w_y + \frac{\partial w_y}{\partial y}dy\right)$$

$$= \rho(x,y)dxdy\frac{\partial^2 w}{\partial t^2}$$

or, after simplifying

$$Q\frac{\partial w_x}{\partial x} + \frac{\partial Q}{\partial x}\left(w_x + \frac{\partial w_x}{\partial x}dx\right) + P\frac{\partial w_y}{\partial y} + \frac{\partial P}{\partial y}\left(w_y + \frac{\partial w_y}{\partial y}dy\right) = \rho(x,y)\frac{\partial^2 w}{\partial t^2}. \tag{5.4}$$

For $dx, dy \to 0$ we have

$$Q\frac{\partial w_x}{\partial x} + \frac{\partial Q}{\partial x}w_x + P\frac{\partial w_y}{\partial y} + \frac{\partial P}{\partial y}w_y = \rho(x,y)\frac{\partial^2 w}{\partial t^2}, \tag{5.5}$$

or

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial w(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[P(x,y)\frac{\partial w(x,y,t)}{\partial y}\right] = \rho(x,y)\frac{\partial^2 w(x,y,t)}{\partial t^2}. \tag{5.6}$$

The above differential equation governs the small transverse vibration of a non-uniform membrane. As usual, the space and time components in (5.6) may be separated by

$$w(x,y,t) = u(x,y)\sin\omega t\,. \tag{5.7}$$

Substituting (5.7) into (5.6) yields the eigenvalue problem for the membrane

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial u(x,y)}{\partial x}\right]+\frac{\partial}{\partial y}\left[P(x,y)\frac{\partial u(x,y)}{\partial y}\right]+\lambda\rho(x,y)=0\,, \tag{5.8}$$

where the eigenvalues λ are the squares of the natural frequencies ω.

The partial differential equation (5.8) may be solved for the unknown eigenvalues λ_{ij} and the corresponding eigenfunctions $u^{(ij)}(x,y)$ if the boundary and initial conditions are given. For example, in the case of a rectangular homogeneous membrane with constant tension T at all points and in all directions and with fixed edges on the boundaries, the solution may be found as

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t+B^{*}_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,, \tag{5.9}$$

where

$$\lambda_{mn}=c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,,\ m,n\ =\ 1,2,\ \ldots\,,\ c=\sqrt{\frac{T}{\rho}}\,. \tag{5.10}$$

If the initial displacement and velocity are given by

$$w(x,y,t)\big|_{t=0}=f(x,y) \tag{5.11}$$

and

$$\frac{\partial w(x,y,t)}{\partial t}\Big|_{t=0} = g(x,y), \tag{5.12}$$

then the coefficients B_{mn} and B^*_{mn} may be found from

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{5.13a}$$

and

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy\,. \tag{5.13b}$$

Equation (5.9), with λ_{mn}, B^*_{mn} and B_{mn} given by (5.10), (5.13a) and (5.13b), represents the closed form solution of the problem.

An analytical solution for the equation (5.8) may be also found in the case of a circular uniform membrane with fixed boundaries.

In our study we will consider a square, per-symmetric, non-uniform membrane, fixed at $x=0$, $x=2a$, $y=0$, $y=2a$. A *per-symmetric* membrane is a membrane whose physical properties are symmetric about the axis $x=a$ and $y=a$, as shown in *Figure* 5-4. It is well known that the mode shapes of a per-symmetric membrane are either *symmetric* or *anti-symmetric*. The symmetric modes can be thus determined by considering one quarter of the membrane only, ie. the area laying between $0<x<a$ and $0<y<a$. The boundary conditions for this membrane are

$$u(0, y) = 0 \tag{5.14a}$$

$$u(x,0) = 0 \tag{5.14b}$$

$$\frac{\partial u(a, y)}{\partial x} = 0 \tag{5.14c}$$

$$\frac{\partial u(x, a)}{\partial y} = 0 \tag{5.14d}$$

In the following section we will analyse the finite difference model for this membrane. We will show that if three symmetric mode shapes, one eigenvalue and the total mass are given then the model, under some conditions that will be determined, yield a unique solution to the inverse problem.

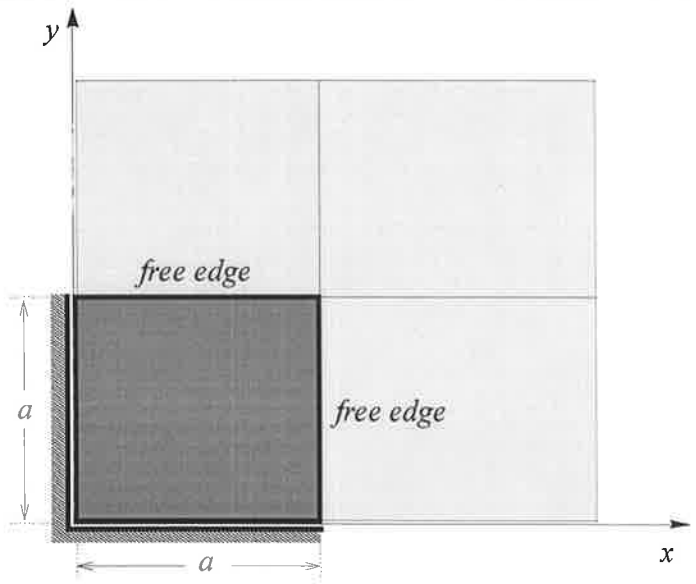

Figure 5-4 A persymmetric membrane

5.3 The finite-difference model

In the method of finite differences, all the derivatives in the differential equation (5.8) and the boundary conditions are replaced by their finite difference approximations. It is important to use the central difference schemes at all nodal points. For simplicity, we assume that the membrane has rectangular geometry with h and k being the grid spacing in x and y - directions, respectively. Also, since the y-coordinate can be scaled, we choose without loss of generality $h=k$.

Let us denote the values of $u(x,y)$ at the grid points $x_i=ih$ and $y_j=jh$, $i, j=1,2,...,n$ by $u_{i,j}$. Using this notation, the differentials in the equation (5.8) may be approximated using the two-point finite difference scheme as

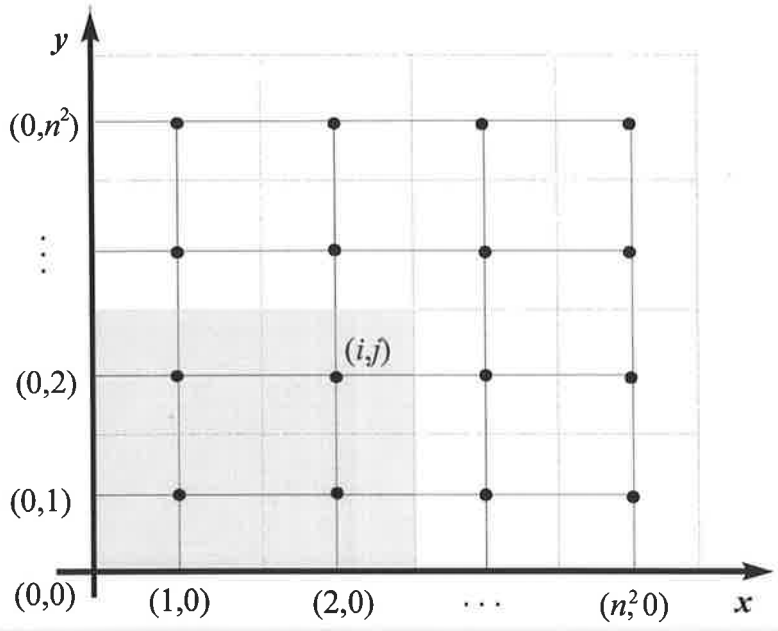

Figure 5-5 The finite difference representation of the membrane

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial u(x,y)}{\partial x}\right]=\frac{1}{h}\left[Q_{i+1/2,j}u'_{i+1/2,j}-Q_{i-1/2,j}u'_{i-1/2,j}\right]+O(h^2)\,. \qquad (5.15)$$

On the other hand, since

$$u'_{i+1/2,j}=\frac{u_{i+1,j}-u_{ij}}{h} \quad \text{and} \quad u'_{i-1/2,j}=\frac{u_{ij}-u_{i-1,j}}{h},$$

we have

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial u(x,y)}{\partial x}\right]=\frac{1}{h^2}\left[Q_{i+1/2,j}(u_{i+1,j}-u_{ij})-Q_{i-1/2,j}(u_{ij}-u_{i-1,j})\right]+O(h^2)\,. \qquad (5.16)$$

In a similar way, we may find

$$\frac{\partial}{\partial y}\left[P(x,y)\frac{\partial u(x,y)}{\partial y}\right]=\frac{1}{h^2}\left[P_{i,j+1/2}(u_{i,j+1}-u_{ij})-P_{i,j-1/2}(u_{ij}-u_{i,j-1})\right]+O(h^2)\,. \qquad (5.17)$$

We may think about the above approximation as a replacement of the continuous membrane by a discrete model consisting of the n^2 masses which interact only between the nearest neighbours by elastic linear strings (see *Figure* 5-5).

Let us denote the masses of a finite difference segment at node (i,j) by m_{ij}, and let the constants of restoring forces in the horizontal and vertical directions be k^H_{ij} and k^V_{ij}, respectively (*Figure* 5-6). We thus define

$$m_{ij}=h^2\,\rho_{ij}, \qquad (5.18)$$

$$k_{ij}^{H} = Q_{i-1/2,j} \tag{5.19}$$

and

$$k_{ij}^{V} = P_{i,j-1/2}. \tag{5.20}$$

Using (5.18) - (5.20) we write (5.16) and (5.17) as

$$\frac{\partial}{\partial x}\left(Q\frac{\partial u}{\partial x}\right) = k_{i+1,j}^{H}\frac{u_{i+1,j}-u_{ij}}{h^2} - k_{ij}^{H}\frac{u_{ij}-u_{i-1,j}}{h^2}, \tag{5.21}$$

and

$$\frac{\partial}{\partial y}\left(P\frac{\partial u}{\partial y}\right) = k_{i,j+1}^{V}\frac{u_{i,j+1}-u_{ij}}{h^2} - k_{ij}^{V}\frac{u_{ij}-u_{i,j-1}}{h^2}. \tag{5.22}$$

Substituting the above expressions into the eigenvalue problem (5.8) yields

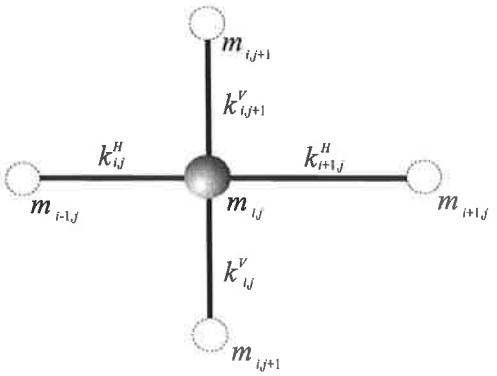

Figure 5-6 The physical parameters of the model

$$k^H_{i+1,j}\frac{u_{i+1,j}-u_{ij}}{h^2}-k^H_{ij}\frac{u_{ij}-u_{i-1,j}}{h^2}+k^V_{i,j+1}\frac{u_{i,j+1}-u_{ij}}{h^2}-k^V_{ij}\frac{u_{ij}-u_{i,j-1}}{h^2}+\lambda\rho_{ij}u_{ij}=0,$$

or

$$k^H_{i+1,j}(u_{i+1,j}-u_{ij})-k^H_{ij}(u_{ij}-u_{i-1,j})+k^V_{i,j+1}(u_{i,j+1}-u_{ij})-k^V_{ij}(u_{ij}-u_{i,j-1})+\lambda m_{ij}u_{ij}=0.$$

Finally, by rearranging the variables in the above equation, we obtain

$$k^H_{ij}u_{i-1,j}-(k^H_{i+1,j}+k^H_{ij})u_{ij}+k^H_{i+1,j}u_{i+1,j}+k^V_{ij}u_{i,j-1}-(k^V_{i,j+1}+k^V_{ij})u_{ij}+k^V_{i,j+1}u_{i,j+1}+\lambda m_{ij}u_{ij}=0,$$

$i=1,2,\ldots,n;\ j=1,2,\ldots,n$ (5.23)

Equation (5.23) is a finite difference representation of the differential equation (5.8).

5.3.1 Boundary conditions

If the membrane is per-symmetric then the associate discrete model is per-symmetric as well. It is thus sufficient to consider one quarter of the membrane only, as shown in *Figure* 5-7. This part of the system may be consider as fixed along the edges $x=0$ and $y=0$, and free along $x=a$ and $y=b$. It should be noted that the free boundary conditions can be imposed by assuming that the masses at the free ends slide on stationary, frictionless rods. These rods supply the tension needed to govern the vibration of the lattice.

In the case of a fixed edge, we require the deflection to vanish along the edge. According to *Figure* 5-7, this yields

$$\text{at } y = 0 : u_{1,0} = u_{2,0} = \cdots = u_{n,0} = 0 \tag{5.24}$$

$$\text{at } x = 0 : u_{0,1} = u_{0,2} = \cdots = u_{0,n} = 0 \tag{5.25}$$

Figure 5-8 shows a node (n,j) at the free edge $x=a$ of the system. At the points $(n+1/2,j)$, $j = 1,2, ..., n$ we require $\frac{\partial u(x,y)}{\partial x} = 0$. This yields

$$\frac{u_{n+1,j} - u_{n,j}}{h} = 0 ; \; x = n+1/2, \; y=1,2, \ldots, n$$

or

$$u_{n+1,j} = u_{n,j} , \; j = 1,2, ..., n. \tag{5.26}$$

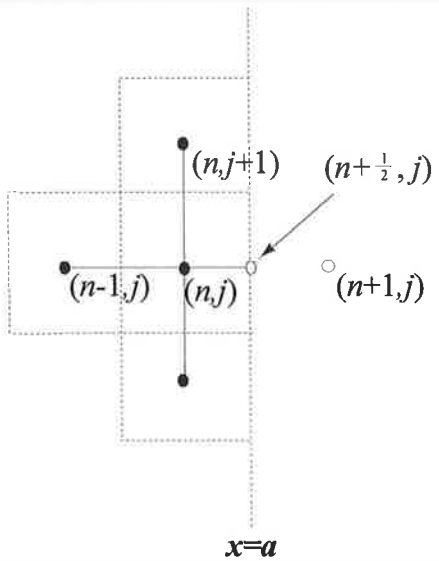

Figure 5-7 The node (n,j) at the free edge $x = a$ of the model

Similarly, along the points $(i,n+1/2)$, $i = 1,2, \ldots, n$ at the free edge $y=b$ we have $\frac{\partial u(x,y)}{\partial y} = 0$, which gives

$$\frac{u_{i,n+1} - u_{i,n-1}}{h} = 0 \; ; x=1,2, \ldots, n, y= n+1/2$$

or

$$u_{i,n+1} = u_{i,n-1} \quad , \; i = 1,2, \ldots, n. \tag{5.27}$$

We note that the points $u_{i,n+1}$ and $u_{n+1,j}$ for $i, j=1,2,\ldots, n$ in the above expressions lie outside of the considered domain. We now show that these non-physical nodes may be eliminated from the system by using equation (5.26) and (5.27). Applying equation (5.23) to points (1,1), (2,1), ...,$(n$-1,1) along the fixed edge $y=0$ yields

$$k_{11}^H u_{01} - (k_{21}^H + k_{11}^H)u_{11} + k_{21}^H u_{21} + k_{11}^V u_{10} - (k_{12}^V + k_{11}^V)u_{11} + k_{12}^V u_{12} + \lambda m_{11} u_{11} = 0 \,,$$

and

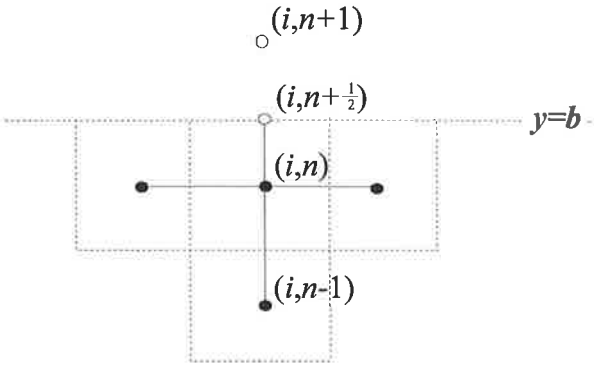

Figure 5-8 The node (i,n) at the free edge $y=b$ of the model

$$k_{i1}^{H}u_{i-1,1}-(k_{i+1,1}^{H}+k_{i,1}^{H})u_{i,1}+k_{i+1,1}^{H}u_{i+1,1}+k_{i,1}^{V}u_{i,0}-(k_{i,2}^{V}+k_{i,1}^{V})u_{i,1}+k_{i,2}^{V}u_{i,2}+\lambda m_{i,1}u_{i,1}=0,$$

where $i=2, 3, ..., n\text{-}1$; $j=1$.

Taking into consideration the boundary conditions (5.24), we have

$$-(k_{21}^{H}+k_{11}^{H})u_{11}+k_{21}^{H}u_{21}-(k_{12}^{V}+k_{11}^{V})u_{11}+k_{12}^{V}u_{12}+\lambda m_{11}u_{11}=0 \qquad (5.28)$$

and

$$k_{i1}^{H}u_{i-1,1}-(k_{i+1,1}^{H}+k_{i,1}^{H})u_{i,1}+k_{i+1,1}^{H}u_{i+1,1}-(k_{i,2}^{V}+k_{i,1}^{V})u_{i,1}+k_{i,2}^{V}u_{i,2}+\lambda m_{i,1}u_{i,1}=0. \qquad (5.29)$$

At the point $(n,1)$ we have

$$k_{n,1}^{H}u_{n-1,1}-(k_{n+1,1}^{H}+k_{n,1}^{H})u_{n,1}+k_{n+1,1}^{H}u_{n+1,1}+k_{n,1}^{V}u_{n,0}-(k_{n,2}^{V}+k_{n,1}^{V})u_{n,1}+k_{n,2}^{V}u_{n,2}+\lambda m_{n,1}u_{n,1}=0.$$

The boundary conditions (5.24) and (5.26) give $u_{n,0}=0$ and $u_{n+1,1}=u_{n,1}$, which yields

$$k_{n,1}^{H}u_{n-1,1}-k_{n,1}^{H}u_{n,1}-(k_{n,2}^{V}+k_{n,1}^{V})u_{n,1}+k_{n,2}^{V}u_{n,2}+\lambda m_{n,1}u_{n,1}=0. \qquad (5.30)$$

Applying equation (5.23) at the points (1,2), (1,3), ...,(1,n-1) along the fixed edge $x=0$, we get

$$k_{1,j}^{H}u_{0,j}-(k_{2,j}^{H}+k_{1,j}^{H})u_{1,j}+k_{2,j}^{H}u_{2,j}+k_{1,j}^{V}u_{1,j-1}-(k_{1,j+1}^{V}+k_{1,j}^{V})u_{1,j}+k_{1,j+1}^{V}u_{1,j+1}$$
$$+\lambda m_{1,j}u_{1,j}=0$$

where $i=1$; $j=2, 3, \ldots, n\text{-}1$,

or equivalently

$$-(k^H_{2,j}+k^H_{1,j})u_{1,j}+k^H_{2,j}u_{2,j}+k^V_{1,j}u_{1,j-1}-(k^V_{1,j+1}+k^V_{1,j})u_{1,j}+k^V_{1,j+1}u_{1,j+1}+\lambda m_{1,j}u_{1,j}=0. \quad (5.31)$$

At the point $(1,n)$ we have

$$-(k^H_{2,n}+k^H_{1,n})u_{1,n}+k^H_{2,n}u_{2,n}+k^V_{1,n}u_{1,n-1}-k^V_{1,n}u_{1,n}+\lambda m_{1,n}u_{1,n}=0, \quad (5.32)$$

and the unknown quantity $k^V_{1,n+1}$ is eliminated from the system.

We now apply equation (5.23) at the points $(n,2)$, $(n,3)$, ..., $(n,n\text{-}1)$ along the free edge $x = a$

$$k^H_{n,j}u_{n-1,j}-(k^H_{n+1,j}+k^H_{n,j})u_{n,j}+k^H_{n+1,j}u_{n+1,j}+k^V_{n,j}u_{n,j-1}-(k^V_{n,j+1}+k^V_{n,j})u_{n,j}+k^V_{n,j+1}u_{n,j+1}$$
$$+\lambda m_{n,j}u_{n,j}=0$$

Taking into consideration the boundary conditions (5.26), we have

$$k^H_{n,j}u_{n-1,j}-k^H_{n,j}u_{n,j}+k^V_{n,j}u_{n,j-1}-(k^V_{n,j+1}+k^V_{n,j})u_{n,j}+k^V_{n,j+1}u_{n,j+1}+\lambda m_{n,j}u_{n,j}=0. \quad (5.33)$$

At the points $(2,n)$, $(3,n)$, ... $(n\text{-}1,n)$ along the free edge $y=b$ we have

$$k^H_{i,n}u_{i-1,n}-(k^H_{i+1,n}+k^H_{i,n})u_{i,n}+k^H_{i+1,n}u_{i+1,n}+k^V_{i,n}u_{i,n-1}-k^V_{i,n}u_{i,n}+\lambda m_{i,n}u_{i,n}=0. \quad (5.34)$$

Finally, at the corner point (n,n) we have

$$k^H_{n,n}u_{n-1,n} - k^H_{n,n}u_{n,n} + k^V_{n,n}u_{n,n-1} - k^V_{n,n}u_{n,n} + \lambda m_{n,n}u_{n,n} = 0. \tag{5.35}$$

As usual, we write the above system in the matrix form

$$\boxed{\mathbf{Ku} = \lambda\mathbf{Mu},} \tag{5.36}$$

where the stiffness matrix **K** has the block tridiagonal form

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & & & \\ \mathbf{B}_1 & \mathbf{A}_2 & \mathbf{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{B}_{n-2} & \mathbf{A}_{n-1} & \mathbf{B}_{n-1} \\ & & & \mathbf{B}_{n-1} & \mathbf{A}_n \end{bmatrix}. \tag{5.37}$$

and the mass matrix **M** is block diagonal

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & & & & \\ & \mathbf{M}_2 & & & \\ & & \ddots & & \\ & & & \mathbf{M}_{n-1} & \\ & & & & \mathbf{M}_n \end{bmatrix}. \tag{5.38}$$

The matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1}$ in (5.37) are tridiagonal and they are given by

$$\mathbf{A}_i = \begin{bmatrix} -(k_{i,1}^H + k_{i+1,1}^H + k_{i,1}^V + k_{i,2}^V) & k_{2,i}^H & & & \\ k_{2,i}^H & -(k_{i,2}^H + k_{i+1,2}^H + k_{i,2}^V + k_{i,3}^V) & k_{3,i}^H & & \\ & \ddots & \ddots & \ddots & \\ & & k_{n-1,i}^H & -(k_{i,n-1}^H + k_{i+1,n-1}^H + k_{i,n-1}^V + k_{i,n}^V) & k_{n,i}^H \\ & & & k_{n,i}^H & -(k_{i,n}^H + k_{i+1,n}^H + k_{i,n}^V) \end{bmatrix}$$

where $i = 1,2, \ldots,(n\text{-}1)$, and $\mathbf{A}_n$ is (5.39a)

$$\mathbf{A}_i = \begin{bmatrix} -(k_{n,1}^H + k_{n,1}^V + k_{n,2}^V) & k_{2,n}^H & & & \\ k_{2,n}^H & -(k_{n,2}^H + k_{n,2}^V + k_{n,3}^V) & k_{3,n}^H & & \\ & \ddots & \ddots & \ddots & \\ & & k_{n-1,n}^H & -(k_{n,n-1}^H + k_{n,n-1}^V + k_{n,n}^V) & k_{n,n}^H \\ & & & k_{n,n}^H & -(k_{n,n}^H + k_{n,n}^V) \end{bmatrix}. \quad (5.39b)$$

The matrices $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{n\text{-}1}$ and $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$ are diagonal. They are given by

$$\mathbf{B}_i = \begin{bmatrix} k_{1,i+1}^V & & & \\ & k_{2,i+1}^V & & \\ & & \ddots & \\ & & & k_{n,i+1}^V \end{bmatrix}, \; i = 1,2,\ldots.n-1 \quad (5.40)$$

and

$$\mathbf{M}_i = \begin{bmatrix} m_{1,i} & & & \\ & m_{2,i} & & \\ & & \ddots & \\ & & & m_{n,i} \end{bmatrix} \; i = 1,2,\ldots, n\,. \quad (5.41)$$

Finally, the eigenvector $\mathbf{u}$ is defined as

$$\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]^{\mathrm{T}}, \quad (5.42)$$

where $\mathbf{u}_i = [u_{i1}\ u_{i2}\ \ldots\ u_{in}]^T$, $i=1,2,\ldots,n$.

The resulting system (5.36) is of order n^2 by n^2, so its non-trivial solution yields n^2 eigenvalues $\lambda_{11}, \lambda_{12}, \ldots \lambda_{n,n}$ and their corresponding eigenvectors $\mathbf{u}^{(11)}, \mathbf{u}^{(12)}, \ldots, \mathbf{u}^{(nn)}$. As in the previous cases, this data may be used to solve the inverse problem of the system. However, if the number n is large, the model may become complex. In the following section we look for an alternative way to represent the system.

5.3.2 An alternative model

The vector $\mathbf{u}$ in eigenvalue problem (5.36) is of dimension n^2 and is not the natural form for a two-dimensional membrane. It is more natural to arrange the mode shape for the membrane in the form

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \tag{5.43}$$

and to rearrange the system (5.36) that includes (5.43) instead of $\mathbf{u}$. To do this we introduce the matrices $\mathbf{A}_i$, $i=1, 2, \ldots, n$ as

$$\mathbf{A}_i = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix}_{(nxn)}, \tag{5.44}$$

and a matrix **E** as

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & \ddots & \ddots & & & \\ & & & & 1 & -1 & \\ & & & & & 1 & \end{bmatrix}_{(nxn)}. \tag{5.45}$$

Then, the system (5.36) may be written as

$$\boxed{\sum_{i=1}^{n} \left\{ \mathbf{A}_i \mathbf{U} (\mathbf{E} \mathbf{K}_i^V \mathbf{E}^T) + (\mathbf{E} \mathbf{K}_i^H \mathbf{E}^T) \mathbf{U} \mathbf{A}_i + \lambda \mathbf{A}_i \mathbf{U} \mathbf{M}_i \right\} = \mathbf{o}}, \tag{5.46}$$

where the stiffness and mass matrices are diagonal n by n matrices of the form

$$\mathbf{K}_i^H = -\begin{bmatrix} k_{1,i}^H & & & \\ & k_{2,i}^H & & \\ & & \ddots & \\ & & & k_{n,i}^H \end{bmatrix}_{(nxn)}, \tag{5.47}$$

$$\mathbf{K}_i^V = -\begin{bmatrix} k_{i,1}^V & & & \\ & k_{i,2}^V & & \\ & & \ddots & \\ & & & k_{i,n}^V \end{bmatrix}_{(nxn)}, \tag{5.48}$$

$$\mathbf{M}_i = \begin{bmatrix} m_{i1} & & & \\ & m_{i2} & & \\ & & \ddots & \\ & & & m_{in} \end{bmatrix}_{(nxn)} . \tag{5.49}$$

For $n=2$ we demonstrate that the matrix system (5.46) is equivalent to the system (5.36). Setting $n=2$ in(5.46) yields

$$\begin{aligned} &\mathbf{A}_1\mathbf{U}(\mathbf{E}\mathbf{K}_1^H\mathbf{E}') + (\mathbf{E}\mathbf{K}_1^V\mathbf{E}')\mathbf{U}\mathbf{A}_1 + \lambda\mathbf{A}_1\mathbf{U}\mathbf{M}_1 \\ &+ \mathbf{A}_2\mathbf{U}(\mathbf{E}\mathbf{K}_2^H\mathbf{E}') + (\mathbf{E}\mathbf{K}_2^V\mathbf{E}')\mathbf{U}\mathbf{A}_2 + \lambda\mathbf{A}_2\mathbf{U}\mathbf{M}_2 = \mathbf{0} \end{aligned} \tag{5.50}$$

Taking into account expressions for A_i, **E** and (5.47)-(5.49), we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -k_{11}^V & \\ & -k_{12}^V \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)$$

$$+\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -k_{11}^H & \\ & -k_{21}^H \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\begin{bmatrix} m_{11} & \\ & m_{12} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -k_{21}^V & \\ & -k_{22}^V \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)$$

$$+\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -k_{12}^H & \\ & -k_{22}^H \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\begin{bmatrix} m_{21} & \\ & m_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$-\begin{bmatrix}(u_{11}(k_{11}^V+k_{12}^V)-u_{12}k_{12}^V) & (-u_{11}+u_{12})k_{12}^V)\\ 0 & 0\end{bmatrix}-\begin{bmatrix}(u_{11}(k_{11}^H+k_{21}^H)-u_{21}k_{21}^H & 0\\ (-u_{11}k_{21}^H+u_{21}k_{21}^H) & 0\end{bmatrix}$$

$$+\lambda\begin{bmatrix}u_{11}m_{11} & u_{12}m_{12}\\ 0 & 0\end{bmatrix}-\begin{bmatrix}0 & 0\\ (u_{21}(k_{21}^V+k_{22}^V)-u_{22}k_{22}^V) & (-u_{21}k_{22}^V+u_{22}k_{22}^V)\end{bmatrix} \tag{5.51}$$

$$-\begin{bmatrix}0 & (u_{12}(k_{12}^H+k_{22}^H)-u_{22}k_{22}^H)\\ 0 & (-u_{12}k_{22}^V+u_{22}k_{22}^V)\end{bmatrix}+\lambda\begin{bmatrix}0 & 0\\ u_{21}m_{21} & u_{22}m_{22}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

The above matrix equation is equivalent to the following four algebraic equations

$$-u_{11}(k_{11}^H+k_{21}^H+k_{11}^V+k_{12}^V)+u_{12}k_{12}^V+u_{21}k_{21}^H+\lambda u_{11}m_{11}=0\,, \tag{5.52a}$$

$$(u_{11}-u_{21})k_{21}^H-u_{21}(k_{21}^V+k_{22}^V)+u_{22}k_{22}^V+\lambda u_{21}m_{21}=0\,, \tag{5.52b}$$

$$(u_{11}-u_{12})k_{12}^V-u_{12}(k_{12}^H+k_{22}^H)+u_{22}k_{22}^H+\lambda u_{12}m_{12}=0 \tag{5.52c}$$

and

$$(u_{12}-u_{22})k_{22}^H+(u_{21}-u_{22})k_{22}^V+\lambda u_{22}m_{22}=0\,. \tag{5.52d}$$

Equations (5.52a)-(5.52d) yield

$$\mathbf{K} = \begin{bmatrix} -(k_{21}^H + k_{11}^H + k_{12}^V + k_{11}^V) & k_{21}^H & k_{12}^V & 0 \\ k_{21}^H & -(k_{22}^H + k_{22}^V + k_{21}^V) & 0 & k_{22}^V \\ k_{12}^V & 0 & (k_{22}^H + k_{12}^H + k_{12}^V) & k_{22}^H \\ 0 & k_{22}^V & k_{22}^H & (k_{22}^H + k_{22}^V) \end{bmatrix}$$

and

$$\mathbf{M} = \begin{bmatrix} m_{11} & & & \\ & m_{21} & & \\ & & m_{12} & \\ & & & m_{22} \end{bmatrix},$$

which is the same as **K** and **M** defined in (5.37) and (5.38).

5.4 The inverse problem

In the inverse problem, the unknowns are the physical parameters k^H_{ij}, k^V_{ij} and m_{ij}, where i,j = 1,2, ...,n, as defined in (5.18)-(5.20). If the membrane is per-symmetric, only one quarter of the model should be considered, since the mode shapes are then either symmetric or anti-symmetric around the edges of the symmetry. In the following, we will analyse the inverse problem for this part of the membrane.

The analytical model of the system is given by (5.36), or alternatively, by (5.46). We note from the *Figure* 5-10 that the mass m_{11} of the finite difference model of the system is fixed to the edges $x=0$ and $y=0$ by two springs k^H_{11} and k^V_{11}. It is thus clear that, in the inverse problem, only the sum $(k^H_{11}+k^V_{11})$ may be determined, not k^H_{11} and k^V_{11} individually. Therefore, in the case of a system of n by n finite difference segments, the total number of unknowns is $(3n^2-1)$. To solve the problem we need to have at least the same number of equations in these unknowns. The eigenvalue problem (5.36) gives n^2 equations if one eigenpair is specified. If three distinct eigenpairs are given, then (5.36) yields $3n^2$ equations. In order to obtain a unique solution we may eliminate one of the eigenvalues from the given data and in this way obtain $3n^2$ equations in $3n^2-1+1=3n^2$ unknowns. However, we know that it is not possible to uniquely reconstruct the system from eigendata only. This is because if $(\lambda,\mathbf{u})$ is one solution of $\mathbf{Ku}=\lambda\mathbf{Mu}$, then $(\lambda,\mathbf{u})$ is

also the solution of $\mathbf{K}_1\mathbf{u}=\lambda\mathbf{M}_1\mathbf{u}$, where $\mathbf{K}_1=\alpha\mathbf{K}$ and $\mathbf{M}_1=\alpha\mathbf{M}$, α - some constant factor.

Therefore, we pose the following problem:

> *Given an eigenvalue* λ_1 *and its corresponding eigenvector* $\mathbf{u}^{(1)}$, *two additional eigenvectors* $\mathbf{u}^{(2)}$ *and* $\mathbf{u}^{(3)}$ and *the total mass W of the lattice. We wish to find the unknown physical properties* k^H_{ij}, k^V_{ij} *and* m_{ij}, $i,j=1,2,\ldots,n$ *of the system which correspond to the given data.*

To solve this problem we apply an iterative method where one of the eigenvalues associated with the eigenvectors $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(3)}$ is used as an iterative parameter. Let λ_2 be the iterative parameter and let the initial value for it be $\lambda_2^{(1)}$, where $\lambda_2^{(1)}\neq\lambda_1$. We start from the last row of the eigenvalue problem (5.36). This gives

$$-\mathbf{B}_{n-1}\mathbf{u}_{n-1}-\mathbf{A}_n\mathbf{u}_n+\lambda\mathbf{M}_n\mathbf{u}_n=\mathbf{0}\,. \tag{5.53a}$$

In the expanded form (5.53a) may be written as

$$\begin{bmatrix} -k^V_{1,n} & & & \\ & -k^V_{2,n} & & \\ & & \ddots & \\ & & & -k^V_{n,n} \end{bmatrix} \begin{Bmatrix} u_{1,n-1} \\ u_{2,n-1} \\ \vdots \\ u_{n,n-1} \end{Bmatrix}$$

$$+\begin{bmatrix} (k^H_{n1}+k^V_{n1}+k^H_{n2}) & -k^H_{n2} & & \\ -k^H_{n2} & (k^H_{n2}+k^V_{n2}+k^H_{n3}) & -k^H_{n3} & \\ & \ddots & \ddots & \\ & & -k^H_{nn} & (k^H_{nn}+k^V_{nn}) \end{bmatrix} \begin{Bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{Bmatrix} \tag{5.53b}$$

$$+\lambda\begin{bmatrix} m_{1n} & & & \\ & m_{2n} & & \\ & & \ddots & \\ & & & m_{nn} \end{bmatrix}\begin{Bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}.$$

The last row of the above system,

$$-k_{nn}^{H}u_{n,n-1} - k_{nn}^{V}u_{n-1,n} + (k_{nn}^{H} + k_{nn}^{V})u_{n,n} + \lambda_1 m_{nn}u_{nn} = 0\,, \tag{5.54}$$

corresponds to the point associated with the mass m_{nn} and also includes the unknown k_{nn}^{H} and k_{nn}^{V}. This equation must be satisfied for all three eigenpairs, ie.

$$-k_{nn}^{H}u_{n,n-1}^{(1)} - k_{nn}^{V}u_{n-1,n}^{(1)} + (k_{nn}^{H} + k_{nn}^{V})u_{n,n}^{(1)} + \lambda_1 m_{nn}u_{nn}^{(1)} = 0\,, \tag{5.55a}$$

$$-k_{nn}^{H}u_{n,n-1}^{(2)} - k_{nn}^{V}u_{n-1,n}^{(2)} + (k_{nn}^{H} + k_{nn}^{V})u_{n,n}^{(2)} + \lambda_2 m_{nn}u_{nn}^{(2)} = 0 \tag{5.55b}$$

and

$$-k_{nn}^{H}u_{n,n-1}^{(3)} - k_{nn}^{V}u_{n-1,n}^{(3)} + (k_{nn}^{H} + k_{nn}^{V})u_{n,n}^{(3)} + \lambda_3 m_{nn}u_{nn}^{(3)} = 0\,. \tag{5.55c}$$

As in the previous cases, the unknown eigenvalue λ_3 may be eliminated from the system by requiring a non-trivial solution for the physical parameters. Consequently, the determinant of the coefficient matrix of (5.55) must vanish. This yields

$$\lambda_3 = \frac{c_2(a_3b_1 - a_1b_3) + c_1(a_2b_3 - a_3b_2)}{c_3(a_2b_1 - a_1b_2)}, \tag{5.56}$$

where the coefficients a_i, b_i and c_i ($i=1,2,3$) are given by

$$a_1 = u_{nn}^{(1)} - u_{n,n-1}^{(1)}; \quad b_1 = u_{nn}^{(1)} - u_{n-1,n}^{(1)}; \quad c_1 = -\lambda_1 u_{nn}^{(1)}, \tag{5.57a}$$

$$a_2 = u_{nn}^{(2)} - u_{n,n-1}^{(2)}; \quad b_2 = u_{nn}^{(2)} - u_{n-1,n}^{(2)}; \quad c_2 = -\lambda_2 u_{nn}^{(2)} \tag{5.57b}$$

and

$$a_3 = u_{nn}^{(3)} - u_{n,n-1}^{(3)}; \quad b_3 = u_{nn}^{(3)} - u_{n-1,n}^{(3)}; \quad c_3 = -u_{nn}^{(3)}. \tag{5.57c}$$

We assume that the system is undamped, so all its eigenvalues must be real and positive. The condition $\lambda_3 > 0$ gives

a) $c_2(a_3b_1 - a_1b_3) > c_1(a_2b_3 - a_3b_2)$,

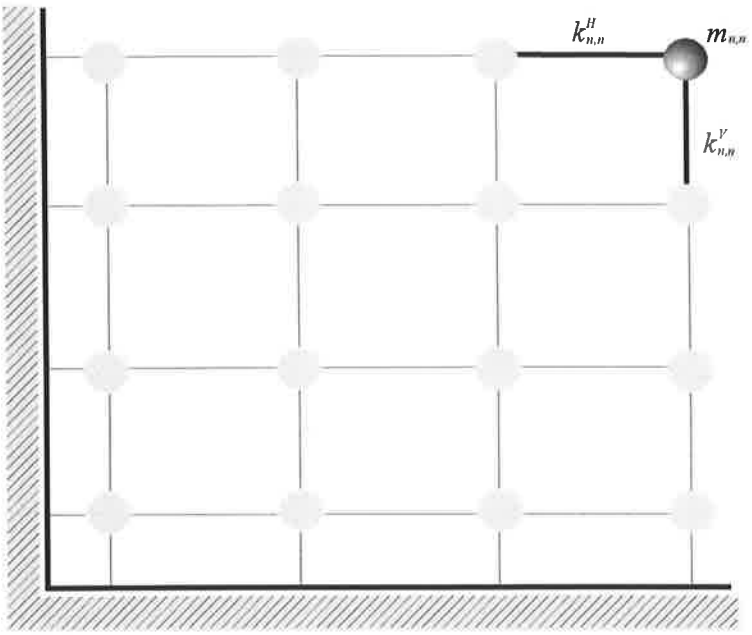

Figure 5-9 The unknowns corresponding to the last row of the eigenvalue problem

if $a_2b_1 > a_1b_2$ and $c_3 > 0$. (5.58a)

b) $c_2(a_3b_1 - a_1b_3) < c_1(a_2b_3 - a_3b_2)$,

if $a_2b_1 > a_1b_2$ and $c_3 < 0$. (5.58b)

c) $c_2(a_3b_1 - a_1b_3) > c_1(a_2b_3 - a_3b_2)$,

if $a_2b_1 < a_1b_2$ and $c_3 > 0$. (5.58c)

e) $c_2(a_3b_1 - a_1b_3) < c_1(a_2b_3 - a_3b_2)$,

if $a_2b_1 < a_1b_2$ and $c_3 < 0$. (5.58d)

Furthermore, we must have

$$a_2b_1 \neq a_1b_2 \text{ and } c_3 \neq 0. \tag{5.58e}$$

Equations (5.58a) - (5.58e) are some of the necessary conditions on the given data to ensure the reconstruction of a physically acceptable system.

Finding the appropriate value for λ_3 we now divide the system (5.55) by m_{nn} and determine the ratios $\hat{k}^H_{nn} = \dfrac{k^H_{nn}}{m_{nn}}$ and $\hat{k}^V_{nn} = \dfrac{k^V_{nn}}{m_{nn}}$ from

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{Bmatrix} \hat{k}^H_{nn} \\ \hat{k}^V_{nn} \end{Bmatrix} = -\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}. \tag{5.59}$$

We note that the condition $\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0$ is already satisfied by (5.58e).

We now consider the (n-1) row of the system (5.53). It follows from *Figure* 5-12 that the set of equations associated with the (n-1) row includes the parameters $m_{n,n-1}$, $k^H_{n,n}$, $k^H_{n,n-1}$ and $k^V_{n,n-1}$. The stiffness $k^H_{n,n}$ is determined in the previous step. Thus, we may write the system in the following form

$$\begin{bmatrix} u^{(1)}_{n,n-1} & (u^{(1)}_{n,n-1} - u^{(1)}_{n-1,n-1}) & (-\lambda_1 u^{(1)}_{n,n-1}) \\ u^{(2)}_{n,n-1} & (u^{(2)}_{n,n-1} - u^{(2)}_{n-1,n-1}) & (-\lambda_2 u^{(2)}_{n,n-1}) \\ u^{(3)}_{n,n-1} & (u^{(3)}_{n,n-1} - u^{(3)}_{n-1,n-1}) & (-\lambda_3 u^{(3)}_{n,n-1}) \end{bmatrix} \begin{Bmatrix} k^H_{n,n-1} \\ k^V_{n,n-1} \\ m_{n,n-1} \end{Bmatrix} = \begin{Bmatrix} (u^{(1)}_{n,n} - u^{(1)}_{n,n-1}) \\ (u^{(2)}_{n,n} - u^{(2)}_{n,n-1}) \\ (u^{(3)}_{n,n} - u^{(3)}_{n,n-1}) \end{Bmatrix} k^H_{nn}. \qquad (5.60)$$

We now divide the system (5.60) by m_{nn} and obtain

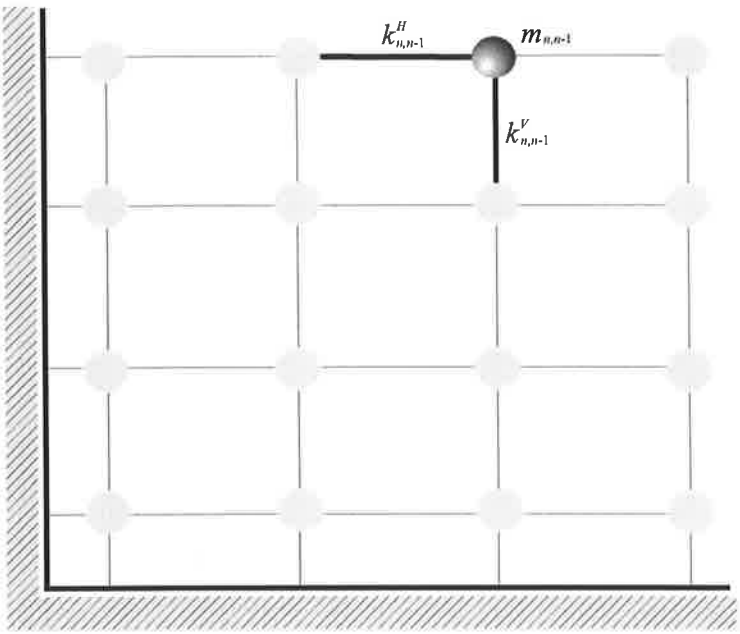

Figure 5-10 The unknowns corresponding to the node (n,n-1) of the system

$$\begin{bmatrix} u^{(1)}_{n,n-1} & (u^{(1)}_{n,n-1}-u^{(1)}_{n-1,n-1}) & (-\lambda_1 u^{(1)}_{n,n-1}) \\ u^{(2)}_{n,n-1} & (u^{(2)}_{n,n-1}-u^{(2)}_{n-1,n-1}) & (-\lambda_2 u^{(2)}_{n,n-1}) \\ u^{(3)}_{n,n-1} & (u^{(3)}_{n,n-1}-u^{(3)}_{n-1,n-1}) & (-\lambda_3 u^{(3)}_{n,n-1}) \end{bmatrix} \begin{Bmatrix} \hat{k}^H_{n,n-1} \\ \hat{k}^V_{n,n-1} \\ \hat{m}_{n,n-1} \end{Bmatrix} = \begin{Bmatrix} (u^{(1)}_{n,n}-u^{(1)}_{n,n-1}) \\ (u^{(2)}_{n,n}-u^{(2)}_{n,n-1}) \\ (u^{(3)}_{n,n}-u^{(3)}_{n,n-1}) \end{Bmatrix} \hat{k}^H_{nn}, \qquad (5.61)$$

where

$$\hat{k}^H_{n,n-1} = \frac{k^H_{n,n-1}}{m_{nn}}, \hat{k}^V_{n,n-1} = \frac{k^V_{n,n-1}}{m_{nn}}, \hat{m}_{n,n-1} = \frac{m_{n,n-1}}{m_{nn}}. \qquad (5.62)$$

The value for $\hat{k}^H_{nn}$ is found in the previous step, so the system (5.61) may be used to evaluate $\hat{k}^H_{n,n-1}$, $\hat{k}^V_{n,n-1}$ and $\hat{m}_{n,n-1}$, if

$$\det \begin{bmatrix} u^{(1)}_{n,n-1} & (u^{(1)}_{n,n-1}-u^{(1)}_{n-1,n-1}) & (-\lambda_1 u^{(1)}_{n,n-1}) \\ u^{(2)}_{n,n-1} & (u^{(2)}_{n,n-1}-u^{(2)}_{n-1,n-1}) & (-\lambda_2 u^{(2)}_{n,n-1}) \\ u^{(3)}_{n,n-1} & (u^{(3)}_{n,n-1}-u^{(3)}_{n-1,n-1}) & (-\lambda_3 u^{(3)}_{n,n-1}) \end{bmatrix} \neq 0. \qquad (5.63)$$

Equation (5.63) is an additional condition on the given data to ensure the reconstruction of a physically acceptable system.

We continue to solve the (n-2), (n-3) ... 1 rows of (5.53) in turn. It follows from *Figure* 5-10 that at the point corresponding to the mass $m_{n,i}$, i=(n-2, n-3, ...,1), the unknown parameters are $m_{n,i}$, $k^H{}_{n,i}$ and $k^V{}_{n,i}$. Introducing $\hat{k}^H_{n,i} = \frac{k^H_{n,i}}{m_{nn}}, \hat{k}^V_{n,i} = \frac{k^V_{n,i}}{m_{nn}}, \hat{m}_{n,i} = \frac{m_{n,i}}{m_{nn}}$ we can determine these parameters from

$$\begin{bmatrix} (u_{n,i}^{(1)}-u_{n,i-1}^{(1)}) & (u_{n,i}^{(1)}-u_{n-1,i}^{(1)}) & (-\lambda_1 u_{n,i}^{(1)}) \\ (u_{n,i}^{(2)}-u_{n,i-1}^{(2)}) & (u_{n,i}^{(2)}-u_{n-1,i}^{(2)}) & (-\lambda_2 u_{n,i}^{(2)}) \\ (u_{n,i}^{(3)}-u_{n,i-1}^{(3)}) & (u_{n,i}^{(3)}-u_{n-1,i}^{(3)}) & (-\lambda_3 u_{n,i}^{(3)}) \end{bmatrix} \begin{Bmatrix} \hat{k}_{n,i}^{H} \\ \hat{k}_{n,i}^{V} \\ \hat{m}_{n,i} \end{Bmatrix} = \begin{Bmatrix} (u_{n,i+1}^{(1)}-u_{n,i}^{(1)}) \\ (u_{n,i+1}^{(2)}-u_{n,i}^{(2)}) \\ (u_{n,i+1}^{(3)}-u_{n,i}^{(3)}) \end{Bmatrix} \hat{k}_{n,i+1}^{H}, \qquad (5.64)$$

assuming that

$$\det \begin{bmatrix} (u_{n,i}^{(1)}-u_{n,i-1}^{(1)}) & (u_{n,i}^{(1)}-u_{n-1,i}^{(1)}) & (-\lambda_1 u_{n,i}^{(1)}) \\ (u_{n,i}^{(2)}-u_{n,i-1}^{(2)}) & (u_{n,i}^{(2)}-u_{n-1,i}^{(2)}) & (-\lambda_2 u_{n,i}^{(2)}) \\ (u_{n,i}^{(3)}-u_{n,i-1}^{(3)}) & (u_{n,i}^{(3)}-u_{n-1,i}^{(3)}) & (-\lambda_3 u_{n,i}^{(3)}) \end{bmatrix} \neq 0. \qquad (5.65)$$

The next step is to consider the (n-1) horizontal line of the model. We start from the point associated with the mass $m_{n\text{-}1,n}$. In a similar way as in the previous step we find $\hat{k}_{n-1,n}^{H}$, $\hat{k}_{n-1,n}^{V}$ and $\hat{m}_{n-1,n}$ using

$$\begin{bmatrix} (u_{n-1,n}^{(1)}-u_{n-1,n-1}^{(1)}) & (u_{n-1,n}^{(1)}-u_{n-2,n}^{(1)}) & (-\lambda_1 u_{n-1,n}^{(1)}) \\ (u_{n-1,n}^{(2)}-u_{n-1,n-1}^{(2)}) & (u_{n-1,n}^{(2)}-u_{n-2,n}^{(2)}) & (-\lambda_2 u_{n-1,n}^{(2)}) \\ (u_{n-1,n}^{(3)}-u_{n-1,n-1}^{(3)}) & (u_{n-1,n}^{(3)}-u_{n-2,n}^{(3)}) & (-\lambda_3 u_{n-1,n}^{(3)}) \end{bmatrix} \begin{Bmatrix} \hat{k}_{n-1,n}^{H} \\ \hat{k}_{n-1,n}^{V} \\ \hat{m}_{n-1,n} \end{Bmatrix} = \begin{Bmatrix} (u_{n-1,n}^{(1)}-u_{n,n-1}^{(1)}) \\ (u_{n-1,n}^{(2)}-u_{n,n-1}^{(2)}) \\ (u_{n-1,n}^{(3)}-u_{n,n-1}^{(3)}) \end{Bmatrix} \hat{k}_{n,n}^{H}, \qquad (5.66)$$

provided that

$$\det \begin{bmatrix} (u_{n-1,n}^{(1)}-u_{n-1,n-1}^{(1)}) & ((u_{n-1,n}^{(1)}-u_{n-2,n}^{(1)}) & (-\lambda_1 u_{n-1,n}^{(1)}) \\ (u_{n-1,n}^{(2)}-u_{n-1,n-1}^{(2)}) & (u_{n-1,n}^{(2)}-u_{n-2,n}^{(2)}) & (-\lambda_2 u_{n-1,n}^{(2)}) \\ (u_{n-1,n}^{(3)}-u_{n-1,n-1}^{(3)}) & (u_{n-1,n}^{(3)}-u_{n-2,n}^{(3)}) & (-\lambda_3 u_{n-1,n}^{(3)}) \end{bmatrix} \neq 0. \qquad (5.67)$$

We may continue to solve the system successively for $i,j=n$-2, n-3, ..., 2, 1 in the same way. At the point corresponding to the mass m_{11} (see *Figure* 5-11), the following system must be satisfied

$$-u_{11}^{(1)}(k_{11}^H + k_{21}^H + k_{11}^V + k_{12}^V) + u_{12}^{(1)}k_{12}^V + u_{21}^{(1)}k_{21}^H + \lambda_1 u_{11}^{(1)}m_{11} = 0, \tag{5.68a}$$

$$-u_{11}^{(2)}(k_{11}^H + k_{21}^H + k_{11}^V + k_{12}^V) + u_{12}^{(2)}k_{12}^V + u_{21}^{(2)}k_{21}^H + \lambda_2 u_{11}^{(2)}m_{11} = 0 \tag{5.68b}$$

and

$$-u_{11}^{(3)}(k_{11}^H + k_{21}^H + k_{11}^V + k_{12}^V) + u_{12}^{(3)}k_{12}^V + u_{21}^{(3)}k_{21}^H + \lambda_3 u_{11}^{(3)}m_{11} = 0. \tag{5.68c}$$

Using (5.68a) and (5.68c) we evaluate $(\hat{k}_{11}^H + \hat{k}_{11}^V)$ and $\hat{m}_{11}$ from

$$\begin{bmatrix} u_{11}^{(1)} & -\lambda_1 u_{11}^{(1)} \\ u_{11}^{(3)} & -\lambda_3 u_{11}^{(3)} \end{bmatrix} \begin{Bmatrix} \hat{k}_{11}^H + \hat{k}_{11}^V \\ \hat{m}_{11} \end{Bmatrix} = \begin{bmatrix} (u_{12}^{(1)} - u_{11}^{(1)}) & (u_{21}^{(1)} - u_{11}^{(1)}) \\ (u_{12}^{(3)} - u_{11}^{(3)}) & (u_{21}^{(3)} - u_{11}^{(3)}) \end{bmatrix} \begin{Bmatrix} \hat{k}_{12}^V \\ \hat{k}_{21}^H \end{Bmatrix}. \tag{5.69}$$

Then we use equation (5.68b) to improve the value for the iterative parameter, ie.

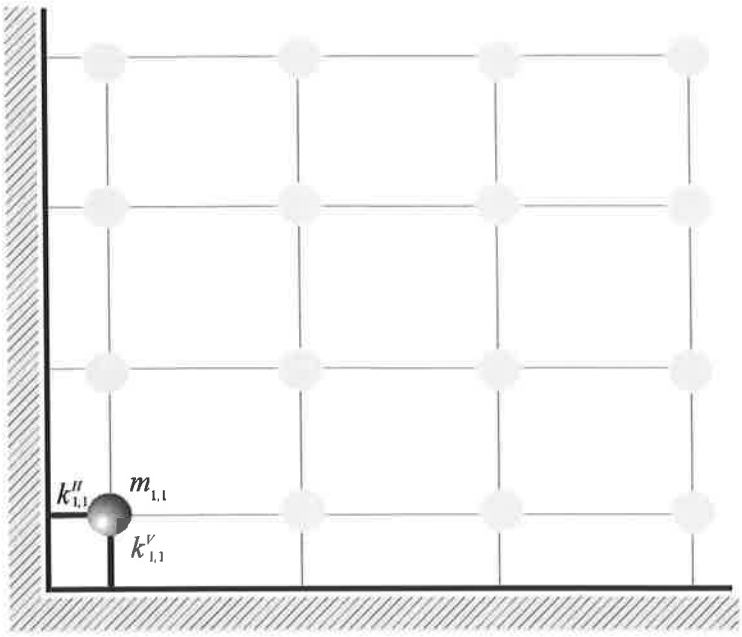

Figure 5-11 The unknowns corresponding to the node (1,1) of the system

$$\lambda_2^{(2)} = -\frac{[u_{21}^{(2)} - u_{11}^{(2)}]\hat{k}_{21}^H + [u_{12}^{(2)} - u_{11}^{(2)}]\hat{k}_{12}^V - [\hat{k}_{11}^H + \hat{k}_{11}^V]u_{11}^{(2)}}{\hat{m}_{11}u_{11}^{(2)}}, \tag{5.70}$$

since($\hat{k}_{11}^H + \hat{k}_{11}^V$) and $\hat{m}_{11}$ are already determined from (5.69). Using $\lambda_2^{(2)}$ we start a new iteration and the process continues in the same way as before. We stop the process when the difference of the values for two successive iterations is less than a given small tolerance, ie. when

$$\frac{\left|\lambda_2^{(i)} - \lambda_2^{(i-1)}\right|}{\lambda_2^{(i)}} < \varepsilon . \tag{5.71}$$

Finally, when the iterations stop, the physical parameters of the system may be evaluated using

$$k_{ij}^H = \frac{W\hat{k}_{ij}^H}{\sum_{i,j}^{n}\hat{m}_{ij}};\ k_{ij}^V = \frac{W\hat{k}_{ij}^V}{\sum_{i,j}^{n}\hat{m}_{ij}};\ m_{ij} = \frac{W\hat{m}_{ij}}{\sum_{i,j}^{n}\hat{m}_{ij}}. \tag{5.72}$$

The above solution is summarised in the following algorithm:

5.4.1 Algorithm 5-1

For an n by n order model, suppose that λ_1, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ and W are given. In order to reconstruct the physical parameters of the system we perform the following steps:

a) Set the starting value for the iterative parameter $\lambda_2^{(1)}$ and the tolerance ε.

b) Evaluate the coefficients a_i, b_i and c_i from (5.57) and λ_3 from (5.56).

c) Solve the system (5.59) and determine $\hat{k}_{nn}^H$ and $\hat{k}_{nn}^V$.

d) Using $\hat{k}_{nn}^H$ determined in the previous step evaluate $\hat{k}_{n,n-1}^H$, $\hat{k}_{n,n-1}^V$ and $\hat{m}_{n,n-1}$ from the system (5.61).

e) Solve the system (5.66) for $i=(n\text{-}2)$, $(n\text{-}3)$, ...1 and evaluate $\hat{k}_{n,i}^H$, $\hat{k}_{n,i}^V$ and $\hat{m}_{n,i}$.

f) Consider the $(n\text{-}1)$, $(n\text{-}2)$, ..., 1 horizontal line of the system and determine $\hat{k}_{i,j}^H$, $\hat{k}_{i,j}^V$ and $\hat{m}_{i,j}$.

g) Evaluate $(\hat{k}_{11}^H + \hat{k}_{11}^V)$ and $\hat{m}_{11}$ from (5.69), and calculate the new value for the iterative parameter $\lambda_2^{(2)}$, using (5.70)

h) Using $\lambda_2^{(2)}$ continue in the same way from step b).

i) Stop the iterations when $\dfrac{\left|\lambda_2^{(i)} - \lambda_2^{(i-1)}\right|}{\lambda_2^{(i)}} < \varepsilon$.

j) Evaluate the physical parameters by solving the system (5.72).

5.5.1 Example 5-1

Suppose that one eigenvalue, three eigenvectors and the total mass of the system from *Figure* 5-15 are given by

$$\lambda_1 = 10 \; ; \mathbf{u}_1 = \{ 0.20 \;\; 0.30 \;\; 0.40 \;\; 0.80 \}^T,$$

$$\mathbf{u}_2 = \{ 0.30 \;\; 0.10 \;\; 0.90 \;\; -0.30 \}^T,$$

$$\mathbf{u}_3 = \{ 0.30 \;\; 0.90 \;\; -0.15 \;\; -0.10 \}^T$$

and

$$W = \sum_{i,j=1}^{n^2} m_{ij} = \; 100.$$

We first set the starting value for the iteration parameter, ie. $\lambda_2^{(1)} = 1$. Using this value we then determine the eigenvalue λ_3 from the equation (5.56), and obtain $\lambda_3 = 218.9205$. Equation (5.59) yields

$$\hat{k}_{22}^{H} = \text{-6.9318 and } \hat{k}_{22}^{V} = 21.5455$$

and similarly by (5.61) we find

$$\hat{k}_{21}^{H} = \text{-30.3478} \; ; \; \hat{k}_{21}^{V} = 60.4031 \text{ and } \hat{m}_{21} = 0.6786.$$

From the point associated with the mass m_{12} we obtain

$$\hat{k}_{12}^{H} = 53.7605 \; ; \; \hat{k}_{12}^{V} = 21.7113 \text{ and } \hat{m}_{12} = 0.3722.$$

At the end, using (5.69) and (5.70) we evaluate

$$\hat{k}_{11}^{H} + \hat{k}_{11}^{V} = 90.6515, \; \hat{m}_{11} = 0.3368 \text{ and } \lambda_2^{(2)} = 16.8804.$$

Using $\lambda_2^{(2)}$ we start a new iteration. Setting $\varepsilon = 0.0001$, the algorithm evaluates, after 35 iterations, $\lambda_2 = 39.3272$, $\lambda_3 = 107.8609$ and the following values for the unknown physical parameters

NODE	k^H_{ij}	k^V_{ij}	m_{ij}
2,2	281.9	509.7	46.0
2,1	91.5	935.7	27.7
1,2	1012.7	691.2	17.9
1,1	763.0	763.0	8.400

Table 5-1. The reconstructed physical parameters of the system

In order to verify this result, the corresponding direct problem has been solved using the data from *Table* 5-1. The resulting values for the first three eigenpairs are given in *Table* 5-2.

NODE	f_1=10.0000	f_2=39.3269	f_3=107.8594
	Φ_1	Φ_2	Φ_3
1,1	0.20	0.30	0.30
1,2	0.30	0.10	0.90
2,1	0.40	0.90	-0.15
2,2	0.80	-0.30	-0.10

Table 5-2.

5.6 An inverse mode problem for a three-dimensional vibrating lattice

Consider the three-dimensional lattice shown in *Figure* 5-13, and assume that each mass vibrates in the z-direction only. This system may be obtained as a generalisation of the two-dimensional lattice by adding the third co-ordinate. The Newton's second law applied on the *i,j,k* mass gives

$$k^H_{ijk}u_{i-1,j,k} - (k^H_{i+1,j,k} + k^H_{ijk})u_{ijk} + k^H_{i+1,j,k}u_{i+1,j,k}$$

$$+ k^V_{ijk}u_{i,j-1,k} - (k^V_{i,j+1,k} + k^V_{ijk})u_{ijk} + k^V_{i,j+1,k}u_{i,j+1,k} \tag{5.73}$$

$$+ k^D_{ijk}u_{i,j,k-1} - (k^D_{i,j,k+1} + k^D_{ijk})u_{ijk} + k^D_{i,j,k+1}u_{i,j,k+1} + \lambda m_{ijk}u_{ijk} = 0,$$

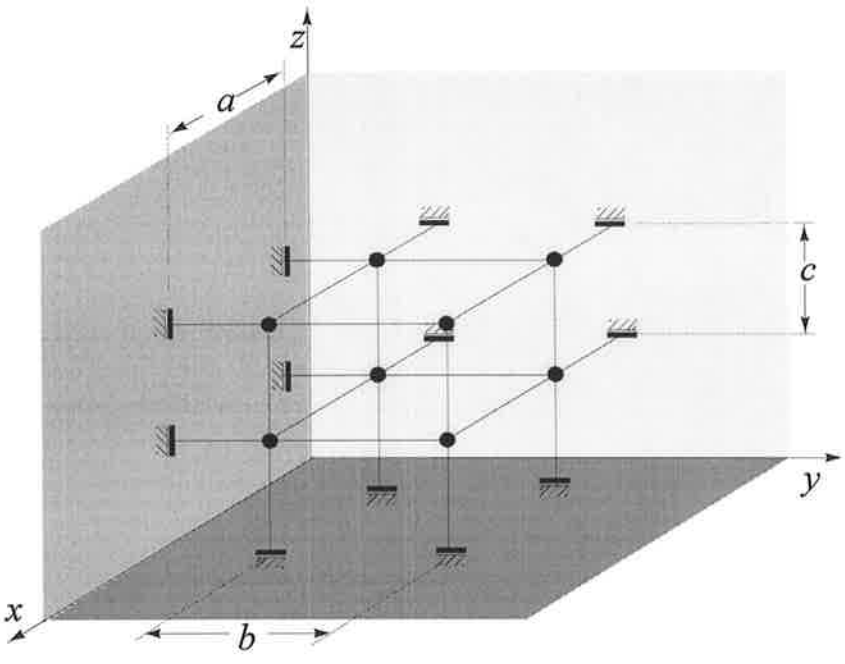

Figure 5-13 A three-dimensional vibrating lattice

where we defined the physical parameters k^H_{ijk}, k^V_{ijk}, k^D_{ijk} and m_{ijk} according to *Figure* 5-14. It should be noted that in (5.73) the parameters k^D_{ijk} are regular stiffnesses of the linear springs, whereas the parameters k^H_{ijk} and k^V_{ijk} are constant tensions of linear strings divided by variable characteristic lengths.

Introducing the fixed boundary conditions along the edges Ox, Oy and Oz and assuming zero slopes at $x=x_1$, $y=y_1$ and $z=z_1$, this system may be written in matrix form as

$$\mathbf{Ku} = \lambda\mathbf{Mu}, \tag{5.74}$$

where **K** and **M** are given by

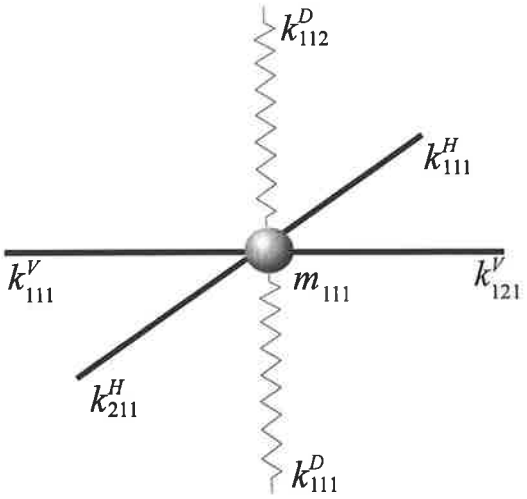

Figure 5-14 The physical parameters at the node (i,j,k)

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & & & \\ \mathbf{B}_1 & \mathbf{A}_2 & \mathbf{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{B}_{n-2} & \mathbf{A}_{n-1} & \mathbf{B}_{n-1} \\ & & & \mathbf{B}_{n-1} & \mathbf{A}_n \end{bmatrix} \tag{5.75}$$

and

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & & & & \\ & \mathbf{M}_2 & & & \\ & & \ddots & & \\ & & & \mathbf{M}_{n-1} & \\ & & & & \mathbf{M}_n \end{bmatrix}. \tag{5.76}$$

The stiffness matrix is block-tridiagonal as in the case of a two-dimensional system. Each submatrix $\mathbf{A}_i$, $i=1, 2, \ldots, n$ is a block-tridiagonal

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{C}_1^{(i)} & \mathbf{D}_1^{(i)} & & & \\ \mathbf{D}_1^{(i)} & \mathbf{C}_2^{(i)} & \mathbf{D}_2^{(i)} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{D}_{n-1}^{(i)} & \mathbf{C}_{n-1}^{(i)} & \mathbf{D}_{n-1}^{(i)} \\ & & & \mathbf{D}_{n-1}^{(i)} & \mathbf{C}_n^{(i)} \end{bmatrix} \tag{5.77}$$

and each submatrix $\mathbf{B}_i$, $i=1, 2, \ldots, n$-1 is diagonal

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{F}_1^{(i)} & & & \\ & \mathbf{F}_2^{(i)} & & \\ & & \ddots & \\ & & & \mathbf{F}_{n-1}^{(i)} \end{bmatrix}. \tag{5.78}$$

In the above expressions $\mathbf{C}_k^{(i)}$, $i,k=1, 2, ..., n$ are tridiagonal, whereas $\mathbf{D}_k^{(i)}$, $i=1, 2, ...,n$; $k=1, 2, ..., n$-1 and $\mathbf{F}_k^{(i)}$, $i,k=1, 2, ..., n$-1 are diagonal matrices.

For example, in the case of a lattice shown in *Figure* 5-13 and taking $k_{ijk}^H = k_{ijk}^V = k_{ijk}^D = k$ and $m_{ijk} = m$, $i,j,k=1,2$ these matrices are given by

$$\mathbf{C}_1^{(1)} = \mathbf{C}_2^{(1)} = \mathbf{C}_1^{(2)} = \mathbf{C}_2^{(2)} = \begin{bmatrix} -6k & k \\ k & -6k \end{bmatrix}, \quad \mathbf{D}_1^{(1)} = \mathbf{D}_1^{(2)} = \begin{bmatrix} k & \\ & k \end{bmatrix},$$

$$\mathbf{F}_1^{(1)} = \mathbf{F}_2^{(1)} = \begin{bmatrix} k & \\ & k \end{bmatrix} \text{ and } \mathbf{N}_1^{(1)} = \mathbf{N}_2^{(1)} = \mathbf{N}_1^{(2)} = \mathbf{N}_2^{(2)} = \begin{bmatrix} m & \\ & m \end{bmatrix}.$$

The stiffness and mass matrices are for this case determined by

$$\mathbf{K} = \begin{bmatrix} -6k & k & k & 0 & k & 0 & 0 & 0 \\ k & -6k & 0 & k & 0 & k & 0 & 0 \\ k & 0 & -5k & k & 0 & 0 & k & 0 \\ 0 & k & k & -4k & 0 & 0 & 0 & k \\ k & 0 & 0 & 0 & -5k & k & k & 0 \\ 0 & k & 0 & 0 & k & -4k & 0 & k \\ 0 & 0 & k & 0 & k & 0 & -4k & k \\ 0 & 0 & 0 & k & 0 & k & k & -3k \end{bmatrix}$$

and

$$\mathbf{M} = \begin{bmatrix} m & & & \\ & m & & \\ & & \ddots & \\ & & & m \end{bmatrix}.$$

5.7 The inverse problem

We now turn our attention to the inverse problem of constructing the physical parameters k^H_{ijk}, k^V_{ijk}, k^D_{ijk} and m_{ijk} from eigendata. We pose the following problem:

> *Given two eigenvalues* λ_1, λ_3, *four eigenvectors* $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ *and* $\mathbf{u}^{(4)}$ *and the total mass W of a three-dimensional lattice. Find the physical parameters of the system* k^H_{ijk}, k^V_{ijk}, k^D_{ijk}, *and* m_{ijk} *that correspond to these dynamic characteristics.*

In order to solve the problem we choose the second eigenvalue as an iteration parameter and set $\lambda_2=\lambda_2^{(1)}$, where $\lambda_2^{(1)}$ is the initial guess. The system (5.74) must be satisfied for all four eigenpairs $(\lambda_i, \mathbf{u}^{(i)})$. Setting $i=n$, $j=n$ and $k=n$, (5.74) yields the following four equations

$$(u^{(1)}_{n-1,n,n}-u^{(1)}_{n,n,n})k^H_{n,n,n}+(u^{(1)}_{n,n-1,n}-u^{(1)}_{n,n,n})k^V_{n,n,n}+(u^{(1)}_{n,n,n-1}-u^{(1)}_{n,n,n})k^D_{n,n,n}+\lambda_1 m_{n,n,n}u^{(1)}_{n,n,n}=0,$$

$$(u^{(2)}_{n-1,n,n}-u^{(2)}_{n,n,n})k^H_{n,n,n}+(u^{(2)}_{n,n-1,n}-u^{(2)}_{n,n,n})k^V_{n,n,n}+(u^{(2)}_{n,n,n-1}-u^{(2)}_{n,n,n})k^D_{n,n,n}+\lambda_2 m_{n,n,n}u^{(2)}_{n,n,n}=0,$$

$$(u^{(3)}_{n-1,n,n}-u^{(3)}_{n,n,n})k^H_{n,n,n}+(u^{(3)}_{n,n-1,n}-u^{(3)}_{n,n,n})k^V_{n,n,n}+(u^{(3)}_{n,n,n-1}-u^{(3)}_{n,n,n})k^D_{n,n,n}+\lambda_3 m_{n,n,n}u^{(3)}_{n,n,n}=0,$$

and (5.79)

$$(u^{(4)}_{n-1,n,n}-u^{(4)}_{n,n,n})k^H_{n,n,n}+(u^{(4)}_{n,n-1,n}-u^{(4)}_{n,n,n})k^V_{n,n,n}+(u^{(4)}_{n,n,n-1}-u^{(4)}_{n,n,n})k^D_{n,n,n}+\lambda_4 m_{n,n,n}u^{(4)}_{n,n,n}=0.$$

Introducing the coefficients

$$a_i = u^{(i)}_{n-1,n,n} - u^{(i)}_{n,n,n},\ b_i = u^{(i)}_{n,n-1,n} - u^{(i)}_{n,n,n},\ c_i = u^{(i)}_{n,n,n-1} - u^{(i)}_{n,n,n},\ i=1,\ 2,\ 3,\ 4$$

and (5.80)

$$d_1 = \lambda_1 u^{(1)}_{n,n,n},\ d_2 = \lambda_2 u^{(2)}_{n,n,n},\ d_3 = \lambda_3 u^{(3)}_{n,n,n},\ d_4 = u^{(4)}_{n,n,n},$$

we write the above system in matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & \lambda_4 d_4 \end{bmatrix} \begin{Bmatrix} k^H_{n,n,n} \\ k^V_{n,n,n} \\ k^D_{n,n,n} \\ m_{n,n,n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.81}$$

The eigenvalue λ_4 may be determined from the system (5.81) by requiring that the determinant of the coefficient matrix vanish (physical parameters must be positive). This gives

$$\lambda_4 = \frac{a_4 D_1 - b_4 D_2 + c_4 D_3}{d_4 D_4}, \tag{5.82}$$

where the minors D_1, D_2, D_3 and D_4 are defined by

$$D_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},\ D_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}, \tag{5.83}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \text{ and } D_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{5.84}$$

Dividing the system (5.81) by $m_{n,n,n}$ yields

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} \hat{k}^H_{n,n,n} \\ \hat{k}^V_{n,n,n} \\ \hat{k}^D_{n,n,n} \end{Bmatrix} = -\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}, \tag{5.85}$$

where

$$\hat{k}^H_{n,n,n} = \frac{k^H_{n,n,n}}{m_{n,n,n}},\ \hat{k}^V_{n,n,n} = \frac{k^V_{n,n,n}}{m_{n,n,n}} \text{ and } \hat{k}^D_{n,n,n} = \frac{k^D_{n,n,n}}{m_{n,n,n}}. \tag{5.86}$$

The system (5.85) may be used to evaluate the parameters $\hat{k}^H_{n,n,n}$, $\hat{k}^V_{n,n,n}$ and $\hat{k}^D_{n,n,n}$ since the coefficients a_i, b_i, c_i and d_i are determined by the given data using (5.80).

The next step is to consider the point corresponding to the mass $m_{n\text{-}1,n,n}$. At this point the following system should be satisfied

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} k^H_{n-1,n,n} \\ k^V_{n-1,n,n} \\ k^D_{n-1,n,n} \\ m_{n-1,n,n} \end{Bmatrix} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} k^H_{n,n,n}, \tag{5.87}$$

where

$$a_i = -u^{(i)}_{n-1,n,n},\ b_i = u^{(i)}_{n-1,n-1,n} - u^{(i)}_{n-1,n,n},\ c_i = u^{(i)}_{n-1,n,n-1} - u^{(i)}_{n-1,n,n}$$

and (5.88)

$$d_i = \lambda_i u^{(i)}_{n-1,n,n} \text{ and } e_i = u^{(i)}_{n-1,n,n} - u^{(i)}_{n,n,n},\ i=1, 2, 3, 4.$$

The system (5.87) determines parameters $\hat{k}^H_{n-1,n,n}$, $\hat{k}^V_{n-1,n,n}$, $\hat{k}^D_{n-1,n,n}$ and $\hat{m}_{n-1,n,n}$, defined as

$$\hat{k}^H_{n-1,n,n} = \frac{k^H_{n-1,n,n}}{m_{n,n,n}},\ \hat{k}^V_{n-1,n,n} = \frac{k^V_{n-1,n,n}}{m_{n,n,n}},\ \hat{k}^D_{n-1,n,n} = \frac{k^D_{n-1,n,n}}{m_{n,n,n}} \text{ and } \hat{m}_{n-1,n,n} = \frac{m_{n-1,n,n}}{m_{n,n,n}}. \quad (5.89)$$

At the point corresponding to the mass $m_{n,n\text{-}1,n}$ we determine in a similar way parameters $\hat{k}^H_{n,n-1,n}$, $\hat{k}^V_{n,n-1,n}$, $\hat{k}^D_{n,n-1,n}$ and $\hat{m}_{n,n-1,n}$. The next step is to consider the point corresponding to the mass $m_{n\text{-}1,n\text{-}1,n}$ (see *Figure* 5-18). At this point the following four equations

$$-u^{(i)}_{n-1,n-1,n}(k^H_{n-1,n-1,n} + k^V_{n-1,n-1,n}) + (u^{(i)}_{n-1,n-1,n-1} - u^{(i)}_{n-1,n-1,n})k^D_{n-1,n-1,n} + (u^{(i)}_{n,n-1,n} - u^{(i)}_{n-1,n-1,n})k^H_{n,n-1,n}$$

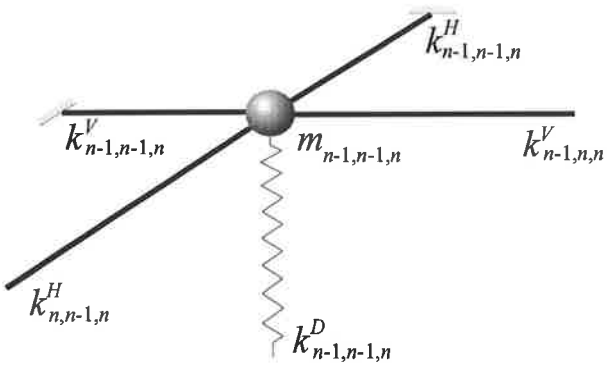

Figure 5-15 The physical parameters at the node (n-1,n-1,n)

$$+ (u^{(i)}_{n-1,n,n} - u^{(i)}_{n-1,n-1,n})k^V_{n-1,n,n} + \lambda_i m_{n-1,n-1,n} u^{(i)}_{n-1,n-1,n} = 0\,,\ i=1,\,2,\,3,\,4 \qquad (5.90)$$

should be used for the determination of $k^H_{n-1,n-1,n}, k^V_{n-1,n-1,n}$ and $m_{n,n\text{-}1,n}$. However, the above system is singular and cannot be used for this purpose. To solve this problem we assume that the mass $m_{n,n\text{-}1,n}$ is connected to the axis Oz by a single string, characterised by the parameter $k^Z_{n-1,n-1,n} = k^H_{n-1,n-1,n} + k^V_{n-1,n-1,n}$ (*Figure* 5-16). Then $\hat{k}^H_{n-1,n-1,n}, \hat{k}^V_{n-1,n-1,n}$, and $\hat{m}_{n-1,n-1,n}$ are determined from the first, the third and the fourth equation of the above system

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{Bmatrix} \hat{k}^Z_{n-1,n-1,n} \\ \hat{k}^D_{n-1,n-1,n} \\ \hat{m}_{n-1,n-1,n} \end{Bmatrix} = \begin{bmatrix} e_1 & f_1 \\ e_3 & f_3 \\ e_4 & f_4 \end{bmatrix} \begin{Bmatrix} \hat{k}^H_{n,n-1,n} \\ \hat{k}^V_{n-1,n,n} \end{Bmatrix}, \qquad (5.91)$$

where

$$a_i = -u^{(i)}_{n-1,n-1,n}\,,\ b_i = u^{(i)}_{n-1,n-1,n-1} - u^{(i)}_{n-1,n-1,n}\,,\ c_i = \lambda_i u^{(i)}_{n-1,n-1,n}\,, \qquad (5.92a)$$

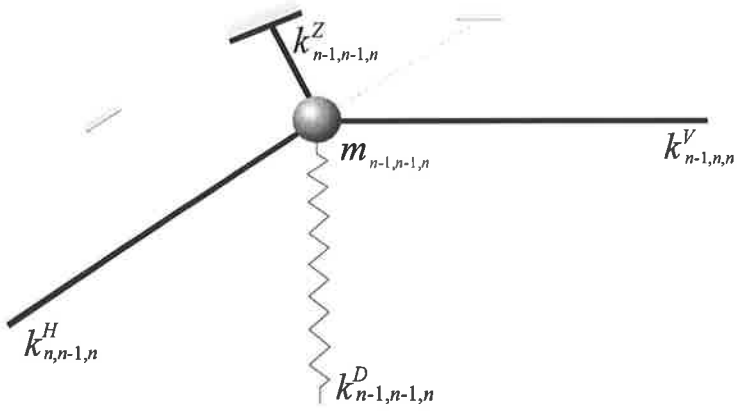

Figure 5-16

$$e_i = u^{(i)}_{n-1,n-1,n} - u^{(i)}_{n,n-1,n} \text{ and } f_i = u^{(i)}_{n-1,n-1,n} - u^{(i)}_{n-1,n,n}, \quad i=1, 3, 4. \tag{5.92b}$$

The remaining equation may be then used to determine the iterative value of λ_2 in the second iteration via

$$\lambda_2^{(2)} = \frac{-a_2\hat{k}^Z_{n-1,n-1,n} - c_2\hat{k}^D_{n-1,n-1,n} + e_2\hat{k}^H_{n,n-1,n} + f_2\hat{k}^V_{n-1,n,n}}{u^{(2)}_{n-1,n-1,n}} \tag{5.93}$$

We continue in the same way and solve the systems corresponding to the mass $m_{n-2,n,n}, \ldots, m_{1,n,n}, \ldots, m_{211}$ successively. At the point corresponding to the mass m_{111}, the following equations must be satisfied

$$-u^{(1)}_{111}(\hat{k}^H_{111} + \hat{k}^V_{111} + \hat{k}^D_{111}) + (u^{(1)}_{211} - u^{(1)}_{111})\hat{k}^H_{211} + (u^{(1)}_{121} - u^{(1)}_{111})\hat{k}^V_{121} + (u^{(1)}_{112} - u^{(1)}_{111})\hat{k}^D_{112} + \lambda_1\hat{m}_{111}u^{(1)}_{111} = 0,$$

$$-u^{(2)}_{111}(\hat{k}^H_{111} + \hat{k}^V_{111} + \hat{k}^D_{111}) + (u^{(2)}_{211} - u^{(2)}_{111})\hat{k}^H_{211} + (u^{(2)}_{121} - u^{(2)}_{111})\hat{k}^V_{121} + (u^{(2)}_{112} - u^{(2)}_{111})\hat{k}^D_{112} + \lambda_2\hat{m}_{111}u^{(2)}_{111} = 0,$$

$$-u^{(3)}_{111}(\hat{k}^H_{111} + \hat{k}^V_{111} + \hat{k}^D_{111}) + (u^{(3)}_{211} - u^{(3)}_{111})\hat{k}^H_{211} + (u^{(3)}_{121} - u^{(3)}_{111})\hat{k}^V_{121} + (u^{(3)}_{112} - u^{(3)}_{111})\hat{k}^D_{112} + \lambda_3\hat{m}_{111}u^{(3)}_{111} = 0$$

and (5.94)

$$-u^{(4)}_{111}(\hat{k}^H_{111} + \hat{k}^V_{111} + \hat{k}^D_{111}) + (u^{(4)}_{211} - u^{(4)}_{111})\hat{k}^H_{211} + (u^{(4)}_{121} - u^{(4)}_{111})\hat{k}^V_{121} + (u^{(4)}_{112} - u^{(4)}_{111})\hat{k}^D_{112} + \lambda_4\hat{m}_{111}u^{(4)}_{111} = 0.$$

The above system is also singular. In order to solve it we assume that the mass m_{111} is connected to the point O by a single spring, characterised by the parameter $k^Z_{111} = k^H_{111} + k^V_{111} + k^D_{111}$.

From the first and the last equation of (5.94) we obtain

$$\begin{bmatrix} -u_{111}^{(1)} & \lambda_1 u_{111}^{(1)} \\ -u_{111}^{(4)} & \lambda_4 u_{111}^{(4)} \end{bmatrix} \begin{Bmatrix} \hat{k}_{111}^{Z} \\ \hat{m}_{111} \end{Bmatrix} = -\begin{bmatrix} (u_{211}^{(1)} - u_{111}^{(1)}) & (u_{121}^{(1)} - u_{111}^{(1)}) & (u_{112}^{(1)} - u_{111}^{(1)}) \\ (u_{211}^{(4)} - u_{111}^{(4)}) & (u_{121}^{(4)} - u_{111}^{(1)}) & (u_{112}^{(4)} - u_{111}^{(4)}) \end{bmatrix} \begin{Bmatrix} \hat{k}_{211}^{H} \\ \hat{k}_{121}^{V} \\ \hat{k}_{112}^{D} \end{Bmatrix}, \tag{5.95}$$

so that $\hat{k}_{111}^{Z}$ and $\hat{m}_{111}$ can be determined. The new values $\lambda_2^{(2)}$ are used in the second iteration and the process continues in the same way. We stop the iterations when

$$\frac{\left|\lambda_2^{(i)} - \lambda_2^{(i-1)}\right|}{\lambda_2^{(i)}} < \varepsilon, \tag{5.96}$$

where ε is a given small tolerance.

Finally, when the iterations stop, we evaluate the physical parameters by using

$$k_{ijk}^{H} = \frac{W\hat{k}_{ijk}^{H}}{\sum_{i,j,k=1}^{n} \hat{m}_{ijk}}; \; k_{ijk}^{V} = \frac{W\hat{k}_{ijk}^{V}}{\sum_{i,j,k=1}^{n} \hat{m}_{ijk}}; \; k_{ijk}^{D} = \frac{W\hat{k}_{ijk}^{D}}{\sum_{i,j,k=1}^{n} \hat{m}_{ijk}} \tag{5.97a}$$

$$k_{ijk}^{Z} = \frac{W\hat{k}_{ijk}^{Z}}{\sum_{i,j,k=1}^{n} \hat{m}_{ijk}} \text{ and } m_{ijk} = \frac{W\hat{m}_{ijk}}{\sum_{i,j,k=1}^{n} \hat{m}_{ijk}}. \tag{5.97b}$$

The above solution is summarised in the following algorithm:

5.7.1 Algorithm 5-2

For an n x n x n lattice, suppose that λ_1, λ_3, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$, $\mathbf{u}^{(4)}$ and the total mass are given. We reconstruct the physical parameters of the system by performing the following steps:

a) Set the starting values for the iterative parameter $\lambda_2^{(1)}$ and the tolerance ε.

b) Evaluate the coefficients a_i, b_i, c and d_i using (5.80).

c) Determine the eigenvalue λ_4 from (5.82) using D_1, D_2, D_3 and D_4 defined by (5.83) and (5.84.

d) Solve the system (5.85) and determine the parameters $\hat{k}^H_{n,n,n}$, $\hat{k}^V_{n,n,n}$ and $\hat{k}^D_{n,n,n}$.

e) Using (5.87), evaluate the parameters $\hat{k}^H_{n-1,n,n}$, $\hat{k}^V_{n-1,n,n}$, $\hat{k}^D_{n-1,n,n}$ and $\hat{m}_{n-1,n,n}$.

f) Determine $\hat{k}^Z_{n-1,n-1,n}$, $\hat{k}^D_{n-1,n-1,n}$ and $\hat{m}_{n-1,n,-1n}$ from (5.91) and find a new value for the iterative parameter $\lambda_2^{(1)}$ from (5.93).

g) For i=n-2, n-3, ..., 2; j=n-2, n-3, ..., 2; and k=n-1, n-2, ..., 2 solve the associate systems successively and determine the corresponding parameters in turn.

h) Using (5.95) find $\hat{k}^Z_{111}$ and $\hat{m}_{111}$ and start a new iteration from the step b).

i) Stop the iterations when $\frac{\left|\lambda_2^{(i)} - \lambda_2^{(i-1)}\right|}{\lambda_2^{(i)}} < \varepsilon$.

j) Evaluate the physical parameters using(5.97a) and (5.97b).

To demonstrate the algorithm we consider 2x2x2 lattice shown in *Figure* 5-20.

5.8 Example 5-2

Let

$$\lambda_1 = 5; \mathbf{u}_1 = \{-0.10\ -0.15\ -0.20\ -0.30\ -0.20\ -0.35\ -0.40\ -0.70\}^T,$$

$$\mathbf{u}_2 = \{\ 0.10\ -0.05\ \ 0.20\ -0.10\ \ 0.30\ -0.15\ \ 0.80\ -0.40\}^T,$$

$$\lambda_3 = 15\mathbf{u}_3 = \{\ 0.10\ \ \ 0.20\ -0.05\ -0.10\ \ 0.40\ \ \ 0.80\ -0.15\ -0.35\}^T,$$

$$\mathbf{u}_4 = \{\ 0.10\ \ 0.30\ \ \ 0.40\ \ 0.85\ -0.05\ -0.10\ -0.15\ -0.30\}^T$$

and

$$W = \sum_{i,j,k=1}^{n^2} m_{ijk} = 50$$

are given. We set the iterative parameters $\lambda_2^{(1)} = 1$ and the tolerance $\varepsilon = 0.00001$. Then, using (5.80 we determine

$$a_1 = u_{122}^{(1)} - u_{222}^{(1)} = 0.30,\ a_2 = u_{122}^{(2)} - u_{222}^{(2)} = 1.20,$$

$$a_3 = u_{122}^{(3)} - u_{222}^{(3)} = 0.20,\ a_4 = u_{122}^{(4)} - u_{222}^{(4)} = 0.15,$$

$$b_1 = u_{212}^{(1)} - u_{222}^{(1)} = 0.35,\ b_2 = u_{212}^{(2)} - u_{222}^{(2)} = 0.25,$$

$$b_3 = u_{212}^{(3)} - u_{222}^{(3)} = 1.15,\ b_4 = u_{212}^{(4)} - u_{222}^{(4)} = 0.20,$$

$$c_1 = u_{221}^{(1)} - u_{222}^{(1)} = 0.40,\ c_2 = u_{221}^{(2)} - u_{222}^{(2)} = 0.30,$$

$$c_3 = u_{221}^{(3)} - u_{222}^{(3)} = 0.25,\ c_4 = u_{221}^{(4)} - u_{222}^{(4)} = 1.15,$$

$$d_1 = \lambda_1 u_{222}^{(1)} = -3.50,\ d_2 = \lambda_2 u_{222}^{(2)} = -0.40$$

and

$$d_3 = \lambda_3 u_{222}^{(3)} = -5.25,\ d_4 = u_{222}^{(4)} = -0.30.$$

The eigenvalue λ_4 is determined using (5.82 which gives $\lambda_4 = 29.9416$. Using (5.85)

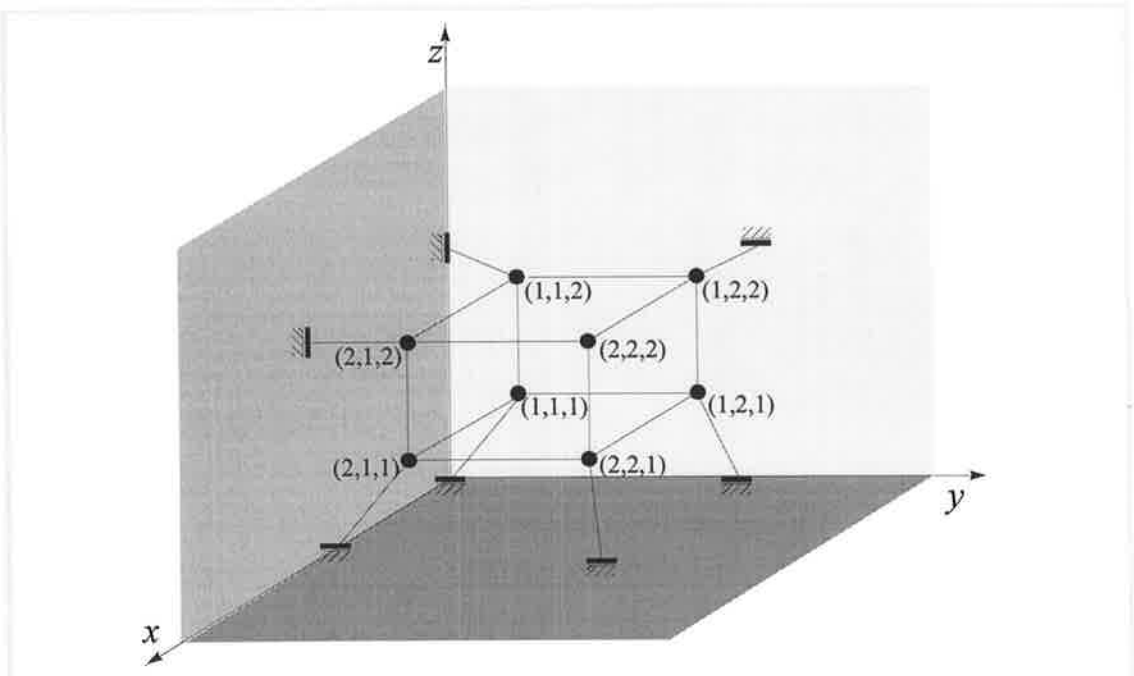

Figure 5-17 The three-dimensional lattice of exaample 5-3

we evaluate the parameters $\hat{k}^H_{222}$, $\hat{k}^V_{222}$ and $\hat{k}^D_{222}$ as

$$\hat{k}^H_{222} = -2.395,\ \hat{k}^V_{222} = 3.3187 \text{ and } \hat{k}^D_{222} = 7.5258.$$

In the next step we consider the node corresponding to the mass m_{122}. At this point the system (5.87) must be satisfied. Its solution gives

$$\hat{k}^H_{122} = -1.5968,\ \hat{k}^V_{122} = 1.8433,\ \hat{k}^D_{122} = 6.1614 \text{ and } \hat{m}_{122} = 0.8170.$$

We continue in the same way and solve the systems corresponding to the mass m_{212}, m_{112} and m_{221}. At the point corresponding to the mass m_{112} we use (5.91) and obtain

$$\hat{k}^Z_{112} = 0.8546,\ \hat{k}^D_{112} = 6.9796 \text{ and } \hat{m}_{112} = 0.6347.$$

Then, using (5.93) we evaluate a new value for the iterative parameter

$$\lambda_2^{(2)} = 1.7190.$$

We continue to solve the systems corresponding to the mass m_{221}, m_{121}, and m_{211}. At the last node (m_{111}) we use (5.95) to obtain

$$\hat{k}^Z_{111} = 13.9427 \text{ and } \hat{m}_{111} = 0.1960.$$

Using $\lambda_2^{(2)}$ we start a new iteration, and the process continue in the same way. After 74 loops the iteration stops and the algorithm produces the values for the physical parameters using $\sum_{i,j,k}^{8} \hat{m}_{ijk} = 5.8134$. The results are shown in *Table* 5-3.

NODE	STIFFNESS				MASS
	$k^H{}_{ijk}$	$k^V{}_{ijk}$	$k^D{}_{ijk}$	$k^Z{}_{ijk}$	m_{ijk}
1,1,1	-	-	-	36.4406	0.6206
2,1,1	25.9610	-	-	55.8326	5.4034
1,2,1	-	41.3230	-	29.5126	4.7241
2,2,1	44.7233	43.7134	46.2729	-	7.8364
1,1,2	-	-	51.9153	30.2113	7.1404
2,1,2	30.4917	32.1350	37.1913	-	7.6982
1,2,2	27.0204	27.8096	33.7044	-	7.9761
2,2,2	23.8622	27.9641	32.8915	-	8.6008

Table 5-3. The reconstructed physical parameters

5.9 Summary

A problem of reconstructing the physical parameters of a two-dimensional per-symmetric membrane and a simply connected three-dimensional lattice has been considered. It has been shown that the physical parameters of the two-dimensional model may be reconstructed if one eigenvalue, three eigenvectors and

the total mass are given. In the case of the three-dimensional lattice, one eigenvalue, four eigenvectors and the total mass are needed for the reconstruction.

In order to determine unknown eigenvalues needed for the solution, we developed an iterative procedure, where these eigenvalues are used as the iterative parameters. The reconstruction algorithms are illustrated for a two and three-dimensional lattices made up of series of masses and elastic rods.

6 Experimental verification

6.1 Introduction

The previous analysis shows that the physical parameters of vibrating systems may be obtained from the knowledge of some of their dynamic properties, ie. the eigenvalues and eigenvectors. The model type and order must be known in advance. If the given data corresponds to the identified model and if the data is complete and error free, the solution is unique. In a practical situation, however, the eigendata must be determined experimentally, for example by the experimental modal analysis. Consequently, this will introduce measurement errors and the results may be corrupted. In order to validate the accuracy of the proposed methods for solving the inverse problems, a simple experiment is carried out. In the following we give the description of the model and experimental set-up.

6.2 The model

An important step in solving the inverse problem is to properly identify the model and its order such that the measured frequencies and mode shapes

realistically correspond to the model. For these reasons a model of a high building structure, as shown in *Figure* 6-4, has been chosen for the experiment. The model is simple, easy to construct and measurements and convenient for re-configuration. The structure is constructed in such a way that its predominant vibrations are in the x direction, so it may be considered as a one dimensional system. It is known from the theory of vibration that such systems may be realistically modelled as the idealised undamped mass-springs systems (*Figure* 6-1). This means that the inverse problem for such systems may be solved applying the solution for the general discrete models, as shown in chapter 3.

The number of degrees-of-freedom of the model corresponds to the number of floors of the structure and it may be easily reconfigured to any desired degrees-of-freedom. The masses m_i of the theoretical model might be approximated by the total weight of the floors. Alternatively, if the weight of the columns is not negligible compare to the weight of floors, they may be taken into account.

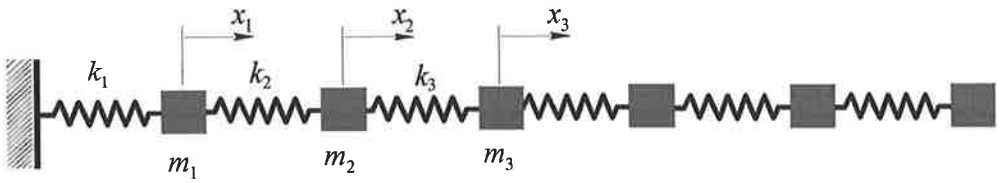

Figure 6-1: The mass-spring model

6.3 The stiffness and mass determination

In the mass-spring system representation of the structure (*Figure* 6-1), k_i are the stiffnesses of the connecting springs associated with the structure's columns. The values for k_i would be equal to the static force that should be applied to the system in the horizontal direction in order to produce a unity deflection.

Applying the methods from the theory of strengths of materials, it can be shown that the stiffnesses of the springs acting in shear are given by

$$k_i = \frac{12(EI)_i}{l^3}, \tag{6.1}$$

where

I - moment of inertia of the spring

E - Young's modulus of elasticity of the spring's material

l - the length of the spring.

However, since there are two sides to each wall, the actual spring's stiffnesses are the sum of the stiffnesses of the individual columns

$$k_i = 2\frac{12(EI)_i}{l^3}, \tag{6.2}$$

assuming that there is no variation in the characteristics between the two sides. On the other hand the moment of inertia of the spring may be given by

$$I_i = \frac{ab^3}{12}, \tag{6.3}$$

where a is the width of the spring and b is the spring's thickness. Substituting (6.3) into (6.2) yields

$$k_i = 2ab^3 \frac{E_i}{l^3}. \tag{6.4}$$

Equation (6.4) may be used for the determination of the theoretical values for the model's stiffnesses. In the actual structure the walls are constructed of aluminium box sections of the length varying from $95mm$ to $100mm$. Substituting these values into (6.4) yields

$$k_i^{(theory)} = 378{,}000N/m - 487{,}000N/m. \tag{6.5}$$

On the other hand, the stiffnesses may be also determined experimentally using the Hooke's law

$$F = -kx, \tag{6.6}$$

where F is the applied force, x is the resulting deflection and k represents the stiffness of the elastic body. The experimental set-up for the stiffness determination of the actual aluminium box sections is shown in *Figure* 6-2. In this experiment, a number of masses M_i are attached onto the box section and the

resulting deflections are measured and recorded. We calculate the stiffness as a ratio between the applied force F_i and the corresponding deflection x_i. The results for one of the box sections are shown in *Table* 6-1. Because of the notable variations in the individual values for the stiffness parameters, we graphically plotted the data (see *Figure* 6-3) and determined the parameters k_i as the slope of the best-fit straight line. Using this method we found that $k_1 = 337550\ N/m$. In a similar way we determined all other stiffness.

The comparison between the experimentally determined values for k_i and the theoretical value (6.5) may be used for the validation of the theoretical model. Once the individual stiffnesses k_i are determined, the stiffness matrix **K** for the whole system may be constructed using the standard three diagonal form

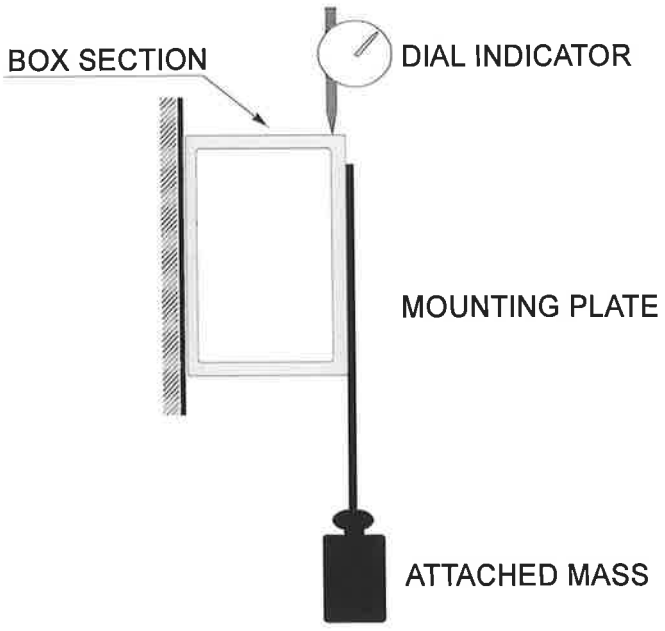

Figure 6-2: The experimental determination of the stiffness

MASS (kg)	FORCE (N)	DEFLECTION (mm)	STIFFNESS (N/m)
2.531	24.8291	0.071	349706
3.120	30.6072	0.087	351807
3.528	34.6097	0.104	332785
4.015	39.3871	0.125	315097
4.587	44.9985	0.135	333322
5.024	49.2854	0.148	333010
5.498	53.9354	0.163	330892
6.005	58.9091	0.171	344497
6.550	64.2555	0.195	329515
7.015	68.8172	0.212	324609
7.495	73.5259	0.225	326782
7.996	78.4408	0.238	329583
8.512	83.5027	0.252	331360
9.012	88.4077	0.272	325028
9.495	93.1459	0.284	327979
9.995	98.0509	0.303	323600
10.510	103.1031	0.307	335841
11.003	107.9394	0.324	333146
11.485	112.6679	0.338	333337
12.007	117.7887	0.351	335580
12.516	122.7820	0.365	336389
12.989	127.4221	0.375	339792
13.508	132.5135	0.392	338045

Table 6-1 Experimental determination of the stiffness k_1

$$\mathbf{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 & & \\ -k_2 & (k_2 + k_3) & -k_3 & \\ & \ddots & \ddots & \\ & & -k_9 & k_9 \end{bmatrix}. \tag{6.7}$$

The mass matrix of the mass-spring model is diagonal and may be either of the form

$$\mathbf{M} = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_9 \end{bmatrix}, \tag{6.8}$$

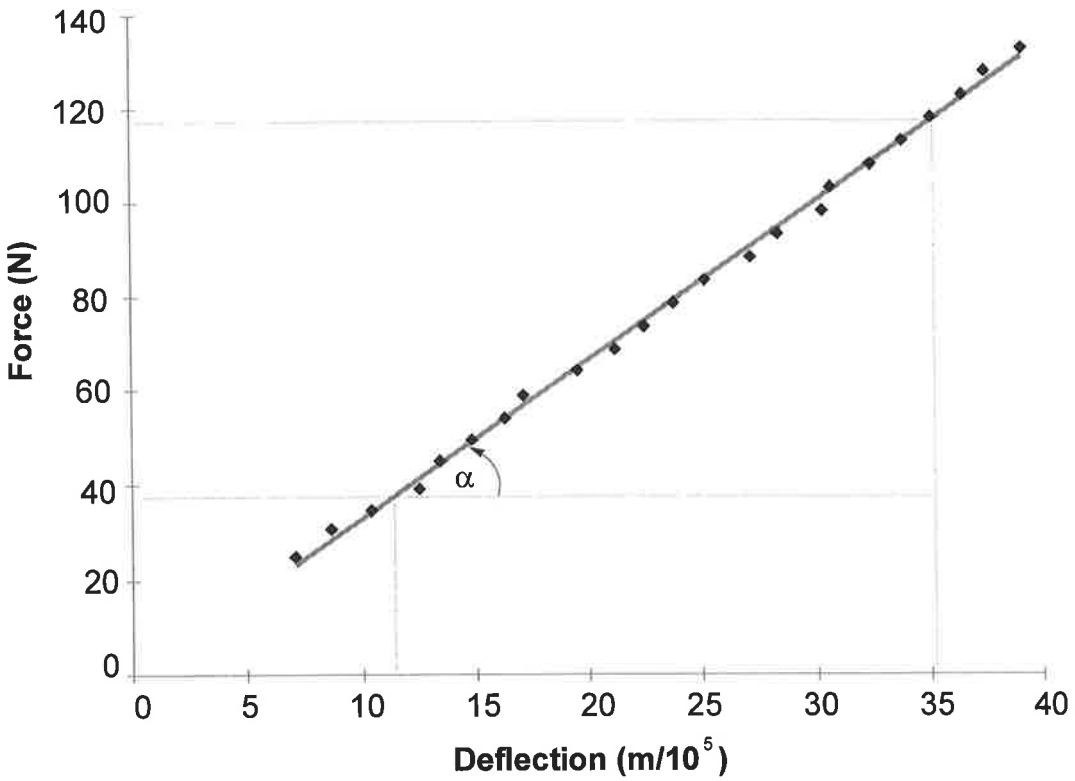

Figure 6-3 The determination of the stiffness k_1

or more accurately as

$$\mathbf{M} = \begin{bmatrix} (M_1 + \frac{2m_1}{3}) & & & \\ & (M_2 + \frac{2m_2}{3}) & & \\ & & \ddots & \\ & & & (M_9 + \frac{2m_{91}}{3}) \end{bmatrix}, \tag{6.9}$$

where the masses of the springs m_i are taken into account. In this model we assume that all masses are rigid, and their values are determined using a laboratory weighing machine. The results are given in *Table* 6-2.

NODE	STIFFNESS (N/m)	MASS M_i (kg)	MASS m_i (kg)
1	337,550	3.916	0. 172
2	352,410	3.408	0. 172
3	350,860	6.897	0. 172
4	353,680	3.556	0. 172
5	347,210	2.702	0. 172
6	345,550	5.876	0. 172
7	353,780	4.313	0. 172
8	351,340	5.841	0.172
9	342,120	4.191	0.086

Table 6-2 The experimental determination of the physical parameters

6.4 Experimental set-up

The actual model of a nine story building structure is shown in *Figure* 6-4. *Figure* 6-6 shows the detail of how plates are connected between the walls and the shape and dimensions of the wall sections. Commercially available aluminium is used for the wall box sections with the average mass of about 0.449 kg. The plates are made from mild and galvanised steel of different thickness. Their approximated masses vary from 0.07kg to 0.5 kg each. The connections between the plates and box sections are obtained by sets of four hex-headed bolts M6x1.0 at each floor.

In order to experimentally obtain the dynamic properties of the model, the structure is excited by means of an impulse hammer at modes 1 to 9. The impact force and model vibrations are measured using a *Brüel and Kjær* accelerometer type 9040 and a *Brüel and Kjær* signal analyser type 2032. The signal produced by model vibrations was amplified by a *Brüel and Kjær* charge amplifier 5666 and analysed on a PC workstation. The schematic detail of the experimental set-up is shown in *Figure* 6-6.

The test procedure consisted of acquiring the response data from a fixed location at the top of the structure, while the impacts were made by the hammer at multiple locations. A prescribed set of excitation points was chosen to coincide with the node points of the model. To obtain the more accurate measurements, these locations were excited ten times each and then arithmetically averaged.

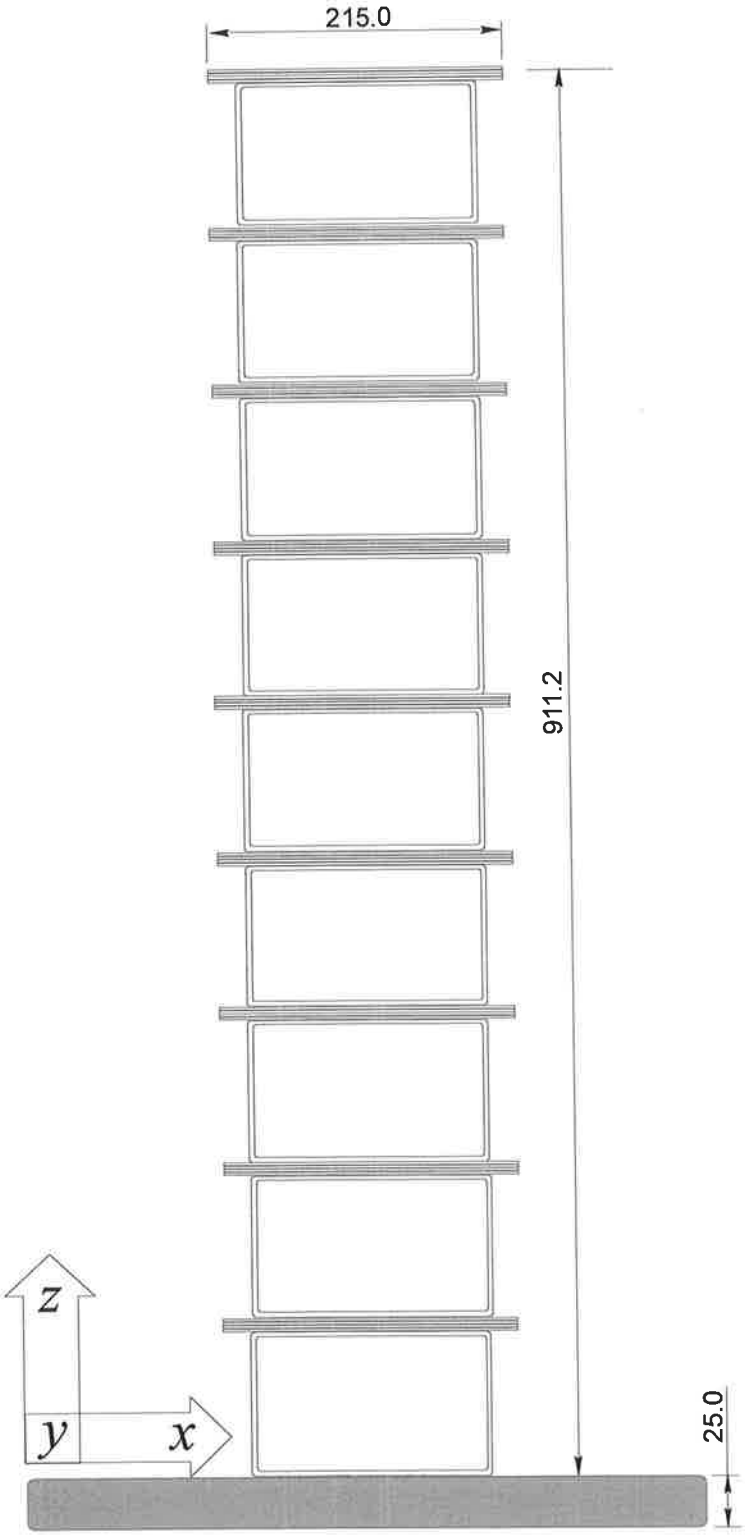

Figure 6-4. The model of a nine story building

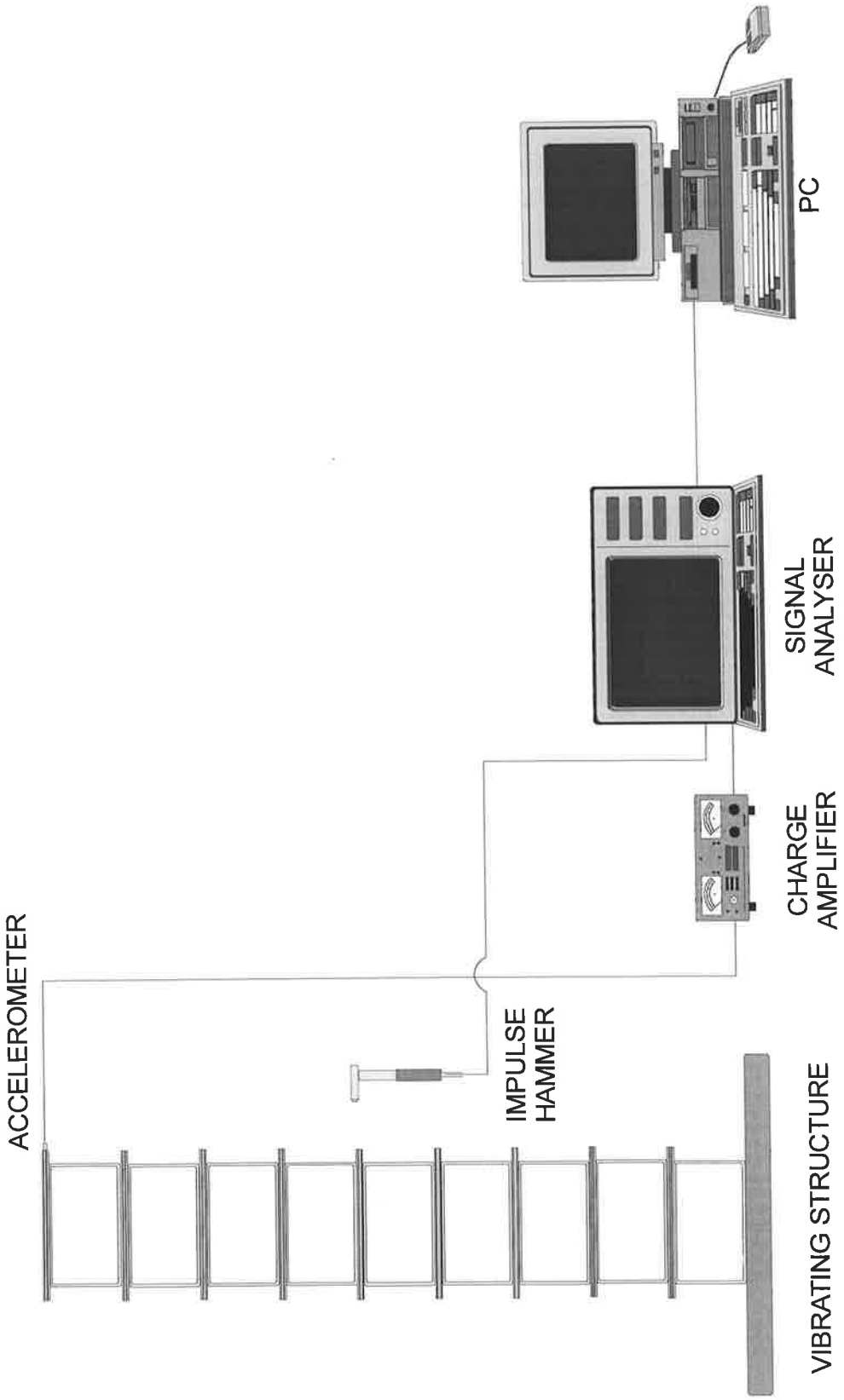

Figure 6-5. The experimental set-up

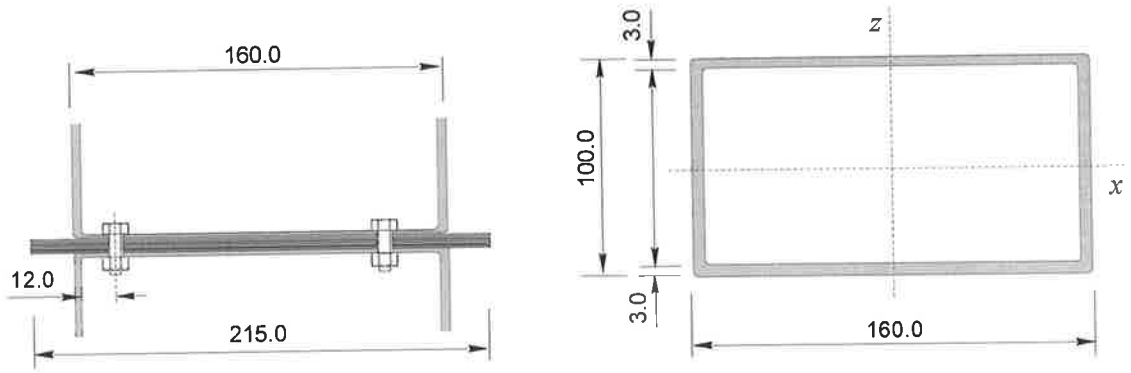

Figure 6-6. The joint and the box section

To obtain experimental resonance frequencies and corresponding mode shapes of the structure, the relative horizontal displacements of each nodal point are measured by an accelerometer. The attached signal analyser transforms the signal from the accelerometer and generates the frequency response function. Since the model acts as a nine-degree-of-freedom discrete system, nine peaks are recorded in a FRF function, as shown in *Figure* 6-7. Using the signal analyser these peaks are accurately identified and their positions measured. This gives us the nine natural frequencies and their associated mode shapes.

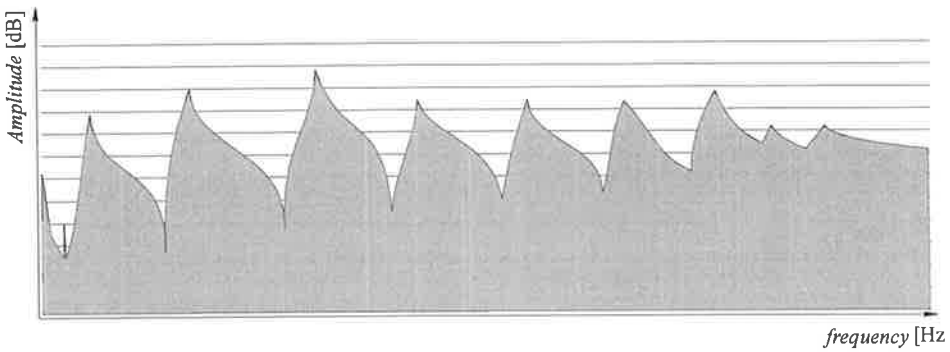

Figure 6-7 The frequency response function

6.5 Experimental results

The values in Hertz for the resonance peaks are given in the *Table* 6-3. The variations in the measurement were relatively small for the frequencies. However, the values for the relative displacements varied by about ±1-3%, depending on the individual modes. We then compared these values with the corresponding values of the theoretical lumped-mass model with **K** and **M** being given by (6.7) and (6.9). The difference in the evaluation of the natural frequency from the theoretical model versus experimental result was less than 7%. Taking into account the variation in the material properties, cross-section and overall dimensions, the theoretical results are acceptable.

MODE	EXPERIMENTAL FREQUENCIES	THEORETICAL FREQUENCIES	RELATIVE ERROR (%)
1	6.75	7.1439	5.51
2	20.50	21.3838	4.13
3	35.25	35.3869	0.39
4	53.00	51.7427	2.43
5	62.00	58.9666	5.14
6	72.75	68.2539	6.59
7	83.00	78.9006	5.19
8	92.25	87.1926	5.80
9	107.00	97.5080	9.73

Table 6-3. The experimental and theoretical values for peak frequencies

6.6 Verification of the inverse algorithm

The next step is to use these values and reconstruct the physical parameters k_i and m_i, $i=1, 2,..., 9$ by solving the inverse problem. In the following we use the algorithm 4.1 developed in Chapter 4. In order to apply the algorithm, one eigenvalue, two eigenvectors and the total mass of the model are needed to be known as the input. All of these properties are measured using the experimental set-up from *Figure* 6-6. An important question here is which two (of possible nine) eigenpairs to use. It has been found that the use of some modes leads to unrealistic results for the reconstructed physical parameters. However, using the third and fourth modes was successful, as will be shown later. To explain this fact we note that the mass-spring representation may give about $n/2$ reasonable accurate eigenvalues and their corresponding eigenvectors. This explains the poor results in applying the higher order modes in the testing. The unrealistic results in using the first two modes may be unexpected, since, theoretically, they should be the most accurate. However, from the physical point of view, it is clear that the lower modes of vibration and in particular the fundamental mode, is influenced by the grounding mass, which is about 25 kg. In the experimental set-up this mass is simply grounded on the floor and therefore may slightly vibrate, while in the theoretical mass-spring model we assume that this end is fixed.

In order to reconstruct the physical parameters k_i and m_i, $i=1, 2, ..., 9$ we apply the algorithm 4.1 for the general discrete vibrating system, developed in Chapter

4. Taking the third and fourth eigenpairs from *Table* 6-4 as input, the algorithm produces the solutions for the physical parameters. The results are given in *Table* 6-5.

We note from *Table* 6-5 that all values for physical parameters are positive. This means that the model may be now physically reconstructed using the definition for the mass and stiffness matrices given by (6.7) and (6.9). The maximum relative error for the calculated values for the stiffnesses is 18.5 % and 15.6 % for the masses. In both cases the maximum errors correspond to the end points of the structure. Clearly, the deviation in the results at these points is a consequence of the errors in the modelling of the boundary conditions.

NODE	f_1=6.75 Hz	f_2=20.50 Hz	f_3=35.25 Hz	f_4=53.00 Hz
	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$
1	105.0	308.5	418.5	242.0
2	98.0	211.5	163.5	-101.0
3	95.5	61.5	-261.5	-288.5
4	87.0	-115.5	-532.0	-108.5
5	68.0	-219.0	-358.5	241.0
6	57.0	-311.5	-46.0	401.0
7	40.0	-346.0	305.0	172.5
8	27.0	-275.5	394.0	-378.0
9	13.0	-170.0	290.0	-595.0

Table 6-4. The first four eigenpairs of the structure

NODE	k_i COMPUTED (N/m)	k_i MEASURED (N/m)	RELATIVE ERROR (%)
1	275085	337550	18.5
2	304276	352410	13.7
3	388336	350860	10.7
4	375573	353680	6.2
5	398246	347210	14.7
6	416117	363550	14.5
7	360598	353780	1.9
8	349022	351340	0.7
9	392700	342120	14.8

NODE	m_i COMPUTED (kg)	m_i MEASURED (kg)	RELATIVE ERROR (%)
1	3.3833	3.9733	14.9
2	3.4255	3.4653	1.1
3	6.5009	6.9553	6.5
4	3.2681	3.3243	1.7
5	2.9714	2.7593	7.7
6	6.5041	5.9333	9.6
7	3.9596	4.3703	9.4
8	6.0092	5.8983	1.9
9	4.8778	4.2197	15.6

Table 6-5. The calculated physical parameters of the model

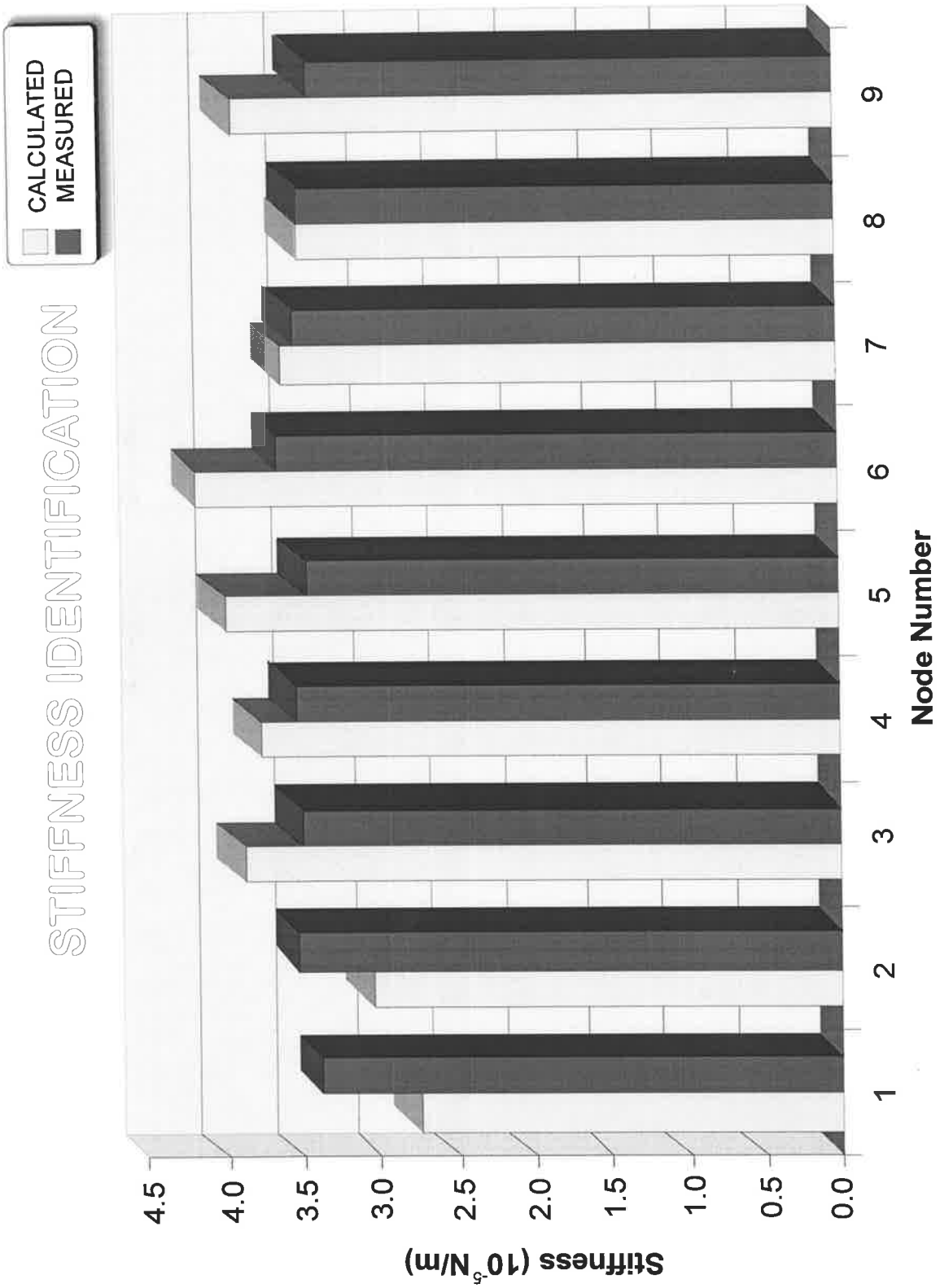

Figure 6-8. The stiffness identification

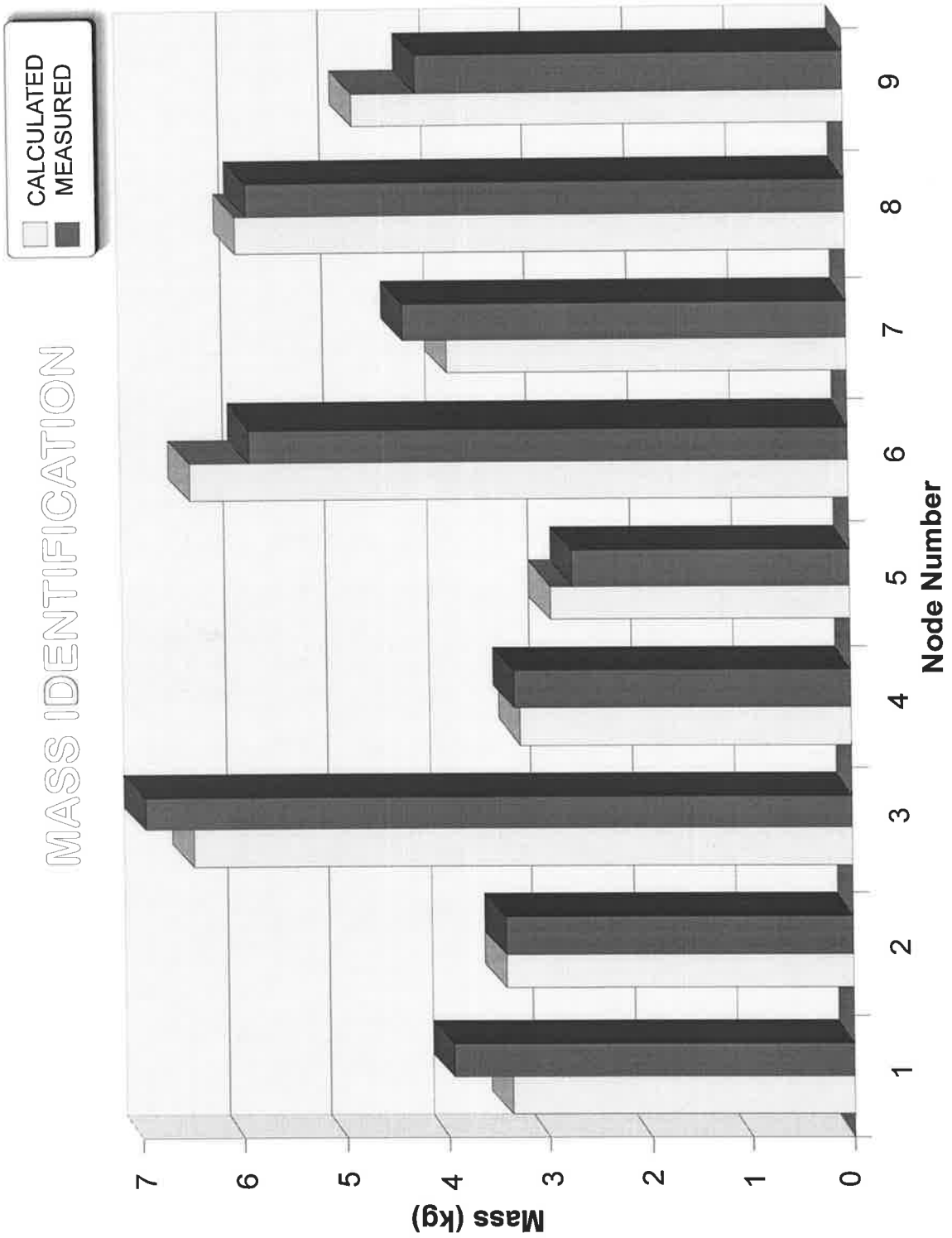

Figure 6-9. The mass identification

6.7 Summary

A model of a higher storey building has been developed and used to experimentally verify the solution of the proposed reconstruction algorithms. The physical parameters of the structure, ie. the masses and stiffnesses of the corresponding mass-spring model, are first experimentally determined by measurements. Then, they are calculated by using the inverse algorithm for the general discrete models, developed in chapter 3. The input data consists of one eigenvalue, two eigenvectors and the total mass of the model. This eigendata are obtained experimentally by exciting the structure by an impact hammer and evaluating the response in number of points, corresponding to the floors of the model.

Comparison between calculated and measured values for the stiffness and the mass parameters shows a good compliance for all particular nodes.

7 Conclusions

In the contemporary engineering analysis a great deal of effort has been invested in the problem of control of structural vibrations. Structural vibration are commonly presented in many machines and products and may decrease the life cycle of structure's parts, increase the level of noise and discomfort and cause breakage and malfunctioning. The main task of the designer is to determine the dynamic behaviour of the system under consideration and to ensure the effects of their structural vibrations are such that the system and its parts operate in a specified range. In order to do this, the test model should be experimentally studied and verified if its vibration response satisfy the specifications. If the results are unfavourable, the initial model must be modified and then tested again. Two different approaches are possible: firstly, we can redesign the initial model by adding (or subtracting) adequate additional elements, and secondly, we may change the physical parameters of the model (masses, stiffnesses or dumping) while keeping the existing configuration and the model order. In both cases, an adequate theoretical model should be defined. The most accurate option is to consider a continuous model based on the distributed physical parameters. Such modelling results in the governing partial differential equations of motion. Taking into

account the initial and boundary conditions the solution of these equation leads to the unlimited number of natural frequencies and their associated mode shapes. However, the problem with such approach is that the analytical solution may be found only in some special cases and for simple mechanical systems. In the case of complex structures with hundreds or thousands of degrees of freedom, the solution may be found only approximately using numerical methods.

The methods of finite differences and finite elements are amongst the most popular of such techniques. This is particularly the case with the finite element method, which becomes widely used as computer technology advances. The major step in applying these methods is to discretize the physical system to a finite number of subsystems - finite differences or finite elements, and in this way to transform the governing differential equations into the set of algebraic equations. Obviously, for a given physical system there may be defined many such models and the basic concern is to determine the most appropriate one. The only practical way to validate the model under consideration is by experimental testing. The modern experimental modal analysis allows a direct measurement of the response properties of the vibrating structures, and in this way we can precisely determine their modal parameters, eg. the natural frequencies and mode shapes. This data may be used to verify the theoretical model. In the case of notable disagreement between the two sets, the initial model must be changed. Assuming that the model structure and the model order are appropriate this implies that the initial values for the physical parameters (masses, stiffnesses and damping) should

be corrected so that the agreement with the test results is improved. This area of science is known as model updating.

In inverse problems the objective is to determine (reconstruct) the physical parameters of the known models without *a priori* knowledge of their initial values. The regular modal analysis approach does not take into consideration the connectivity between the elements of the model and therefore cannot be used for the determination of its physical parameters. The data that is used for the reconstruction consists of the eigenvalues and eigenvectors of the model. Unlike the inverse eigenvalue approach, which is based on the knowledge of all of the eigenvalues of the model, we use here both, the eigenvalues and eigenvectors. It has been shown that the systems with known connectivity may be reconstructed if a small number of eigenvalues and their associated eigenvectors are known. This data may be accurately determined from the frequency response functions obtained by experimental modal analysis.

In chapter 3. we used this approach to solve the inverse problem for a higher order finite difference model of an axially vibrating rod. Ram and Gladwell [71] have solved the problem for the finite element model of the rod, where the stiffness and the mass matrices are both tridiagonal and symmetric. It is shown that the finite difference model of the rod may also be reconstructed using similar methodology. In this model, the standard technique is to apply the two-point finite difference schemes, which are characterised by the errors of order $O(h^2)$, h being the length of an element. In the inverse approach, which are generally ill-posed problems, this

is often not good. In order to improve the accuracy of the model we introduced a higher order finite difference scheme based on four-point approximation for the derivatives. Introducing the higher order approximation in the differential equation for the rod resulted in a diagonal mass matrix as in the standard model and in a five diagonal stiffness matrix, which is generally non-symmetric. Although this introduces some difficulties into the analysis, the inverse problem is then successfully solved. It has been shown that the mass and the stiffness matrices may be reconstructed using one eigenvalue, two eigenvectors and the total mass of the system. If this data corresponds to the analytical model, the solution is unique. In the presence of noise, however, which is more likely to happen in a real situation, the solution may not be realistic or physically realisable. In order to solve this problem and reduce the sensitivity of the numerical procedure we used more eigenpairs in the analysis. The additional data resulted in an overdetermined system, which has been solved in an optimal sense by using the least square method.

In chapter 4 we develop a method for the solution of the inverse problem for general discrete models of vibrating systems. This may be a finite difference, a finite element, or a lumped-mass model of one-dimensional systems such as mass-spring systems, rods or beams. The stiffness and mass matrices appearing in the eigenvalue problem $\mathbf{K}\Phi = \lambda\mathbf{M}\Phi$, are represented as the unspecified functions of the physical parameters. It has been shown that these parameters may be reconstructed from the knowledge of one eigenvalue, two eigenvectors and the

total mass of the system. Since the actual form of **K** and **M** is not needed to be specified, the reconstruction algorithm is applicable as a general solution. The first step in the procedure is to evaluate the eigenvalue λ_2 that is associated with the given eigenvector $\mathbf{u}^{(2)}$. In order to solve this problem the original system of order $2n$ is reduced to an n order eigenvalue problem for which λ_2 is one of the eigenvalues. The reduced eigenvalue problem is characterised by the n dimensional vector $\mathbf{a}=[\alpha_1\ \alpha_2\ \ldots\ \alpha_n]$, where $\alpha_i>0$, which may be regarded as an eigenvector of the system. It is well known that the jth eigenvector of a Jacobian matrices has exactly $(j\text{-}1)$ sign reversals, which means that there is only one eigenvector $\mathbf{a}^{(k)}$ that has all positive components. The eigenvalue $\lambda_2{}^{(k)}$ associated with this particular eigenvector is then the required second eigenvalue of the system. Using this value, the unknown physical parameters are found by taking into consideration the knowledge of the total mass and finding the inverse of the resulting coefficient matrix. This general solution is then applied to some particular vibrating systems. It is demonstrated by means of numerical examples that the solution may be applied for the case of finite difference and finite element models of non-uniform rods and beams, and for discrete models of multi-storey buildings.

In chapter 5 the problem of reconstructing the physical parameters has been posed and solved for linear two and three-dimensional vibrating systems, such as non-homogeneous membranes and lattices. It has been shown that in the case of two-dimensional systems, the data needed for the solution consists of one eigenvalue,

three eigenvectors and the total mass. In the case of a three-dimensional lattice one eigenvalue, four eigenvectors and the total mass is needed to be known. The presented reconstruction algorithm is demonstrated on two numerical examples. It is shown that if the given data corresponds to the analytical model for which the inverse problem is solved, the solution is unique and exact.

In order to validate the practical applicability and behaviour of the proposed methodology, an appropriate experiment is carried out. A model of a multi-storey building, which may be accurately modelled as a discrete mass-spring system, is chosen for the test. The dynamic characteristics of the model, ie. the natural frequencies and their corresponding mode shapes are accurately determined from the frequency response function obtained by an experimental modal analysis equipment. Using certain two eigenpairs corresponding to these data we applied the reconstruction algorithm developed in chapter 3 of the thesis. Then, we compared the produced values for the mass and stiffness parameters with those obtained by measurements. Using the eigendata corresponding to the third and fourth modes gives excellent results, which indicates the practical applicability of the proposed methodology.

Future Directions of Research

There are several problems which remain open. The analysis of chapter 4 has been made for the case where the known data of the finite element model consist of displacements. In several models however, the model is based on the

displacements and slopes (eg. finite element model of the vibrating beam). An appropriate analysis is needed to generalise the results of this chapter to encounter this case. In chapter 5 we have shown how to reconstruct the model of two and three dimensional lattices from the knowledge of their eigenvalues and eigenvectors. The important problem of reconstructing such a model from eigenvalues only is still needed to be addressed. We leave these problems for further investigation in the future.

REFERENCES

1. H. Ahmedian, G. M. L. Gladwell and F. Ismail, Finite element model identification using modal data, Journal of Sound and Vibration **172**, pp. 657-669 (1994).

2. V. Barcilon, On the solution of the inverse problem with amplitude and natural frequency data I, Phys. Earth Planet. Int. **13**, pp. 1-8 (1976).

3. V. Barcilon, On the multiplicity of solutions of the inverse problems for a vibrating beam, SIAM J. Appl. Math. **37**, pp. 119-127 (1979).

4. V. Barcilon, Inverse problem for the vibrating beam in the free-clamped configuration, Phil. Trans. R. Soc. Lond. A **304**, pp. 211-251 (1982).

5. V. Barcilon, A two-dimensional inverse eigenvalue problem, Inverse problems **6**, pp. 11-20 (1990).

6. A. Berman, Mass matrix correction using an incoplete set of meausered modes, AIAA Journal, Vol. **9** pp. 1147-1148 (1979).

7. A. Berman, Comment on optimal weighted orthogonalization of meausere modes, AIAA Journal, Vol. **17** (1979).

8. A. Berman, Improvement of a large analytical model using test data, AIAA Journal, Vol. **21** (1983).

9. A. Berman, System identification of structural dynamic models - theoretical and pratical bounds, AIAA Paper 84-0929, pp. 123-129 (1984).

10. D. Boley and G. H. Golub, A survey of matrix inverse eigenvalue problems, Inverse problems **3**, pp. 595-622 (1987).

11. C. de Boor and H. G. Golub, The numerically stable reconstruction of a Jacobi matrix from spectral data, Linear Alg. Appl. **21**, pp. 245-260 (1978).

12. G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Acta Mathematica **78**, pp. 1-96 (1946).

13. S. G. Braun and Y. M. Ram, Predicting the effect of structural modifications: Upper and lower bounds due to modal truncation, Int. Journal of Anal. and Exp. Modal Analysis **6**, pp. 199-211 (1991).

14. M. T. Chu, Solving additive inverse eigenvalue problems for symmetric matrices by the homotopy method, IMA J. Numer. Anal. **9**, pp. 331-342 (1990).

15. M. T. Chu, Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations, SIAM J. Math. Anal. **22**, pp. 1050-1060 (1991).

16. M. T. Chu, Constructing a Hermitian matrix from its diagonal entries and eigenvalues, SIAM J. Math. Anal. Appl. **16**, pp. 207-217 (1995).

17. B. N. Datta, S. Elhay and Y. M. Ram, Orthogonality and partial pole assignment for the symmetric definite quadratic pencil, Linear Algebra Appl., to appear.

18. C. Davini, A. Morassi and N. Rovere, Modal analysis of notched bars-tests and comments on the sensitivity of an identification procedure, Journal of Sound abd Vibration **179**, pp. 513-527 (1995).

19. A. C. Dowing and A. S. Householder, Some inverse characteristics value problems, J. Assoc. Comput. Mach. **3**, pp. 203-207 (1956).

20. S. Friedland, Inverse eigenvalue problems, Linear Alg. Appl. **17**, pp. 15-71 (1977).

21. S. Friedland, The reconstruction of a symmetric matrix from the spectral data, Journal of Math. Anal. Appl. **71**, pp. 412-422 (1979).

22. S. Friedland, J. Nocedal, and M. L. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, SIAM J. Numer. Anal. **24**, pp. 634-337 (1987).

23. F. P. Gantmakher and M. G. Krein, Oscillation Matrices and Kernels and Small Vibration of Mechanical Systems, State Publishing House for Technical-Theoretical Literature, Moscow-Leningrad, USSR (1950).

24. I. M. Gel'fand and B. M. Levitan, On the determination of a differential equation from its spectral function, Amer. Math. Soc. Transl. **1**, pp. 253-304 (1951).

25. G. M. L. Gladwell, The inverse problem for vibrating beam, Proc. Roy. Soc. Lond. A **393**, pp. 277-295 (1984).

26. G. M. L. Gladwell and J. A. Gbadeyan, On the inverse problem of the vibrating string or rod, Q. J. Mech. Appl. Math. **38**, pp. 169-174 (1985).

27. G. M. L. Gladwell, Inverse Problems in Vibration, Martinus Nijhoff, Dordrecht (1986).

28. G. M. L. Gladwell, The inverse mode problem for lumped-mass systems, Q. J. Mech. Appl. Math. **39**, pp. 297-307 (1986).

29. G. M. L. Gladwell, The inverse problem for the Euler-Bernoulli beam, Proc. Roy. Soc. Lond. A **407**, pp. 199-218 (1986).

30. G. M. L. Gladwell and S. R. A. Dods, Examples of reconstruction of vibrating rods from spectral data, Journal of Sound and Vibration **119(2)**, pp. 267-276 (1987).

31. G. M. L. Gladwell, A. H. England and D. Wang, Examples of reconstruction of an Euler-Bernoulli beam from spectral data, Journal of Sound and Vibration **119**, pp. 81-94 (1987).

32. G. M. L. Gladwell, N. B. Willms, B. He and D. Wang, How can we recognise an acceptable mode shape for a vibrating beam ?, Q. J. Mech. Appl. Math. **42**, pp. 303-316 (1989).

33. G. M. L. Gladwell, Qualitative properties of finite element models I: Sturm-Liouville systems, Q. Journal Mech. Appl. Math. **44**, pp. 249-265 (1991).

34. G. M. L. Gladwell, Qualitative properties of finite element models II: the Euler-Bernoulli beam, Q. Journal Mech. Appl. Math. **44**, pp. 267-284 (1991).

35. G. M. L. Gladwell and B. R. Zhu, Inverse problems for multidimensional vibrating systems, Proc. Roy. Soc. Lond. A **439**, pp. 511-530 (1992).

36. G. M. L. Gladwell and M. Movahhedy, Reconstruction of a mass-spring system from spectral data I: theory, Inverse Problems in Engineering **1**, pp. 179-189 (1995).

37. C. Gordon, D. Webb and S. Wolpert, One cannot hear the shape of a drum, Bull. Amer. Math. Soc. **27**, pp. 134-138 (1992).

38. H. P. W. Gottlieb, Hearing the shape of an annular drum, Journal Austral. Math. Soc. B **24**, pp. 435-438 (1983).

39. H. P. W. Gottlieb, Multi-segment strings with exactly harmonic spectra, Jpurnal of Sound and Vibration **118**, pp. 283-290 (1987).

40. H. P. W. Gottlieb, Inhomogeneous annular plates with exactly beam-like radial spectra, IMA Journal Appl. Math. **50**, pp. 107-112 (1993).

41. O. H. Hald, Inverse eigenvalue problems for Jacobi matrices, Linear Algebra Appl. **14**, pp. 63-85 (1976).

42. O. H. Hald, The Sturm-Liouville problem with symmetric potentials, Acta Math. **141**, pp. 263-291 (1978).

43. O. H. Hald and J. R. McLaughlin, Solutions of inverse nodal problems, Inverse Problems **5**, pp. 307-347 (1989).

44. H. Hochstadt, On some inverse problems in matrix theory, Arch. Math. **18**, pp. 201-207 (1967).

45. H. Hochstadt, On the construction of a Jacobi matrix from spectral data, Linear Algebra Appl. **8**, pp. 435-446 (1974).

46. H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, SIAM J. Appl. Math. **34**, pp. 315-342 (1978).

47. M. Kac, Can one hear the shape of a drum?, Amer. Math. Monthly **73**, pp. 1-23 (1966).

48. P. Lancester and J. Maroulas, Inverse eigenvalue problems for damped vibrating systems, J. Math. Anl. Appl. **123**, pp. 238-261 (1987).

49. B. M. Levitan, Inverse Sturm-Liouville problems, VNU Science Press, Utrecht Netherlends (1987).

50. W. Lihua and W. Shangwen, A method of structural dynamic modification for high building, Proceedings of the 10th Intern. Modal Analysis Conference , Vol II, pp. 1148-1151 (1992).

51. B. D. Lowe, On the construction of an Euler-Bernoulli beam from spectral data, Journal of Sound and Vibration, **163**, pp. 165-171 (1993).

52. B. D. Loewe and W. Rundell, An inverse problem for a Sturm-Liouville operator, Journal Math. Anal. **181**, pp. 188-199 (1994).

53. M. P. Mattis ans H. Hochstadt, On the construction of band matrices from spectral data, Linear Algebra Appl. **38**, pp. 109-119 (1981).

54. J. R. McLaughlin, Analytical methods for recovering coefficients in differential equations from spectral data, SIAM Rev. **28**, pp. 53-72 (1986).

55. J. R. McLaughlin, O. Hald, A formula for finding a potential from nodal lines, Bull. AMS, New Series **32**, pp. 241-247 (1995).

56. J. E. Mottershead and M. I. Friswell, Model updating in structural dynamics: a survey, Journal of Sound and Vibration **167(2)**, pp. 347-375 (1993).

57. M. Movahhedy, F. Ismail and G. M. L. Gladwell, Reconstruction of a mass-spring system from spectral data II: experiment, Inverse Problems in Engineering **1**, pp. 315-327 (1995).

58. G. N. de Oliviera, Nonnegative matrices with prescribed spectrum, Linear Algebra Appl. **54**, pp. 117-121 (1983).

59. J. Paine, A numerical method for the inverse Sturm-Liouville problem, SIAM J. Sci. Stat. Comput. **5**, pp. 149-4-156 (1984).

60. R. Prosser, Can one see the shape of a surface?, Amer. Math. Monthly **84**, pp. 259-270 (1977).

61. M. Protter, Can one hear the shape of a drum? Revisited, SIAM Rev. **29**, pp. 185-197 (1987).

62. Y. M. Ram, S. Braun and J. J. Blech, Structural modification in truncated systems by the Rayleigh-Ritz method, Journal of Sound and Vibration **125**, pp. 203-209 (1988).

63. Y. M. Ram, J. J. Blech and S. G. Braun, Eigenproblem error bounds with application to the symmetric dynamic system modification, SIAM Journal of Matrix Analysis Appl. **11**, pp. 553-564 (1990).

64. Y. M. Ram and S. G. Braun, Upper and lower bounds for the natural frequencies of modified structures based on truncated modal testing results, Journal of Sound and Vibration **137**, pp. 69-81 (1990).

65. Y. M. Ram and S. G. Braun, Structural dynamic modification using truncated data: bounds for the eigenvalues, Mech. Systems Signal Processing **4**, pp. 39-52 (1990).

66. Y. M. Ram and S. G. Braun, An inverse problem associated with the dynamic modification of structures, Journal of Appl. Mech. **58**, pp. 233-237 (1991).

67. Y. M. Ram and S. G. Braun, Eigenvalue error bounds and their applications to structural modification, AIAA J **31**, pp. 759-764 (1993).

68. Y. M. Ram and J. Caldwell, Physical parameters reconstruction of a free-free mass-spring system from its spectra, SIAM Journal of Appl. Math. **52**, pp. 140-152 (1992).

69. Y. M. Ram, An inverse mode problem for the continuous model of an axially vibrating rod, Journal of Appl. Mech. **61**, pp. 624-628 (1994).

70. Y. M. Ram, Inverse mode problems for the discrete model of a vibrating beam, Journal of Sound and Vibration **169**, pp. 239-252 (1994).

71. Y. M. Ram and G. M. L. Gladwell, Constructing a finite element model of a vibrating rod from eigendata, Journal of Sound and Vibration **169**, pp. 229-237 (1994).

72. Y. M. Ram and S. Elhay, The construction of band symmetric models for vibrating systems from modal analysis data, Journal of Sound and Vibration **181**, pp. 583-594 (1995).

73. W. Rundell and P. E. Sacks, The reconstruction of Sturm-Liouville operators, Inverse Problems **8**, pp. 457-482 (1992).

74. W. Rundell and P. E. Sacks, Reconstruction techniques for classical inverse Sturm-Liouville problems, Math. Comp. **58**, pp. 161-183 (1992).

75. L. Starek and D. J. Inman, On the inverse vibration problem with rigid-body modes, ASME Journal of Appl. Mech. **58**, pp. 1101-1104 (1991).

76. L. Starek and D. J. Inman, Inverse problem in vibration for generating symmetric coefficient matrices, Vibration Analysis - Anal. and Comp. ASME **37**, pp. 13-17 (1991).

77. L. Starek, D. J. Inman and A. Kress, A symmetric inverse vibration problem, Journal of Appl. Mech. **114**, pp. 564-568 (1992).

78. B. R. Zhu, K. R. Jackson and R. P. K. Chan, Inverse spectrum problems for block Jacobi matrix, Journal Comp. Math. **11**, pp. 313-322 (1993).

APPENDIX A.

$$k_{11} = -616E_1A_1 + 72E_2A_2 - 8E_3A_3$$

$$k_{12} = 148E_1A_1 + 63E_2A_2 - 7E_3A_3$$

$$k_{13} = -4E_1A_1 - 9E_2A_2 + E_3A_3$$

$$k_{21} = 63E_1A_1 + 204E_2A_2 - 72E_3A_3 + 9E_4A_4$$

$$k_{22} = -360E_2A_2$$

$$k_{23} = -56E_1A_1 + 192E_2A_2 + 64E_3A_3 + 9E_4A_4$$

$$k_{24} = 7E_1A_1 - 12E_2A_2 - 8E_3A_3 + E_4A_4$$

$$k_{i,i} = -360E_iA_i$$

$$k_{i,i+2} = -(E_{i-2}A_{i-2} - 8E_{i-1}A_{i-1} + 12E_iA_i + 8E_{i+1}A_{i+1} - E_{i+2}A_{i+2})$$

$$k_{i,i+1} = -8k_{i,i+2} + 96E_iA_i$$

$$k_{i,i-1} = 8k_{i,i+2} + 288E_iA_i$$

$$k_{i,i-2} = -k_{i,i+2} - 24E_iA_i$$

$$k_{n-1,n} = 7E_{n-3}A_{n-3} - 56E_{n-2}A_{n-2} + 180E_{n-1}A_{n-1} + 49E_nA_n$$

APPENDIX A.

$$k_{n-1,n-1} = -360E_{n-1}A_{n-1}$$

$$k_{n-1,n-2} = -(8/7)(k_{n-1,n} - 348E_{n-1}A_{n-1})$$

$$k_{n-1,n-3} = (1/7)(k_{n-1,n} - 264E_{n-1}A_{n-1})$$

$$k_{n,n} = E_{n-2}A_{n-2} - 72E_{n-1}A_{n-1} - 104\,E_nA_n$$

$$k_{n,n-1} = -9(k_{n,n}/8 + E_nA_n)$$

$$k_{n,n-2} = k_{n,n}/8 + 9E_nA_n\,.$$

APPENDIX B.

$$F = \frac{144}{n^2}$$

$$a_1 = -(8u_1 + 7u_2 - u_3)$$

$$a^*_1 = -(8u^*_1 + 7u^*_2 - u^*_3)$$

$$b_1 = 154u_1 - 37u_2 + u_3$$

$$b^*_1 = 154u^*_1 - 37u^*_2 + u^*_3$$

$$a_2 = 9u_1 - 8u_3 + u_4$$

$$a^*_2 = 9u^*_1 - 8u^*_3 + u^*_4$$

$$b_2 = 17u_1 - 30u_2 + 16u_3 - u_4$$

$$b^*_2 = 17u^*_1 - 30u^*_2 + 16u^*_3 - u^*_4$$

$$a_i = u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}$$

$$a^*_i = u^*_{i-2} - 8u^*_{i-1} + 8u^*_{i+1} - u^*_{i+2}$$

$$b_i = u_{i-2} - 16u_{i-1} + 30u_i - 16u_{i+1} + u_{i+2}$$

$$b^*_i = u^*_{i-2} - 16u^*_{i-1} + 30u^*_i - 16u^*_{i+1} + u^*_{i+2}$$

APPENDIX B.

$$a_{n-1} = u_{n-3} - 8\,u_{n-2} + 7\,u_n$$

$$a^*_{n-1} = u^*_{n-3} - 8\,u^*_{n-2} + 7\,u^*_n$$

$$b_{n-1} = u_{n-3} - 16\,u_{n-2} + 30\,u_{n-1} - 15\,u_n$$

$$b^*_{n-1} = u_{n-3} - 16\,u^*_{n-2} + 30\,u^*_{n-1} - 15\,u^*_n$$

$$a_n = u_{n-2} - 9\,u_{n-1} + 8\,u_n$$

$$a^*_n = u^*_{n-2} - 9\,u^*_{n-1} + 8\,u^*_n$$

$$b_n = u_{n-2} - 27\,u_{n-1} + 26\,u_n$$

$$b^*_n = u^*_{n-2} - 27\,u^*_{n-1} + 26\,u^*_n$$