Dian-Feng Zhang, Xi-Ming Sun, Chong-Quan Zhong, Zhao-Jing Wu, Peng Shi
Noise-to-state stability for random affine systems with state-dependent switching
Proceedings of the 11th World Congress on Intelligent Control and Automation, 2014 / pp.2191-2195

DOI: In Press

Noise-to-state stability for random affine systems with state-dependent switching *

Dian-Feng Zhang
School of Control Science and Engineering
Dalian University of Technology
Dalian, 116024, P.R. China

Zhao-Jing Wu
School of Mathematics and Informational Science
Yantai University
Yantai, 264005, P.R. China

Xi-Ming Sun
School of Control Science and Engineering
Dalian University of Technology
Dalian, 116024, P.R. China
sunxm@dlut.edu.cn

Peng Shi
School of Electrical and Electronic Engineering
University of Adelaide
SA 5005, Australia
College of Engineering and Science
Victoria University
Melbourne, VIC 8001, Australia

Chong-Quan Zhong
School of Control Science and Engineering
Dalian University of Technology
Dalian, 116024, P.R. China

***Abstract*— In this paper, noise-to-state stability is investigated for a class of random systems with state-dependent switching. Under some mild and easily verified conditions, the existence of global solution to random switched systems can be proved by the aid of Lyapunov approach. Based on a reasonable requirement to the random disturbance, the criteria on noise-to-state stability in probability of random switched systems are presented by applying single Lyapunov function technique.**

***Index Terms*— Random affine systems, random switched systems, noise-to-state stability**

I. INTRODUCTION

In practice, control systems are very often affected by noise. As it is known, the definition of deterministic input-to-state stability (ISS) for nonlinear systems was proposed by E.D.Sontag in [1]. Some equivalent manners: dissipation, robustness margins, "$\beta+\gamma$" estimates and classical Lyapunov-like definitions were presented for the deterministic nonlinear systems, see for instance [2], [3]. Recently, ISS has become a central concept and plays an important role in the nonlinear systems analysis. The notion of noise-to-state stability (NSS) was proposed for stochastic nonlinear systems by regarding the unknown covariance of Brown motion as the deterministic input in [4] and [5]. The concept of input-to-state practical stability (ISpS) in probability was given in [6] with respect to a deterministic input. To describe the stochastic stability under stochastic inputs, a more practical notion of stochastic input-to-state stability (SISS) with respect to a stochastic input was introduced in [7]. NSS in the m-th moment and in probability to random nonlinear systems were investigated in [8], where the random signals are stochastic processes whose second-order moments are finite (or bounded).

On the other hand, stability analysis of switched systems has been a major issue in recent years. It is well-known that the switched system is uniformly stable for an arbitrary switching law if there exists a common Lyapunov function for all subsystems in [9], [10]. In particular, ISS as an important property was extend to the switched systems in [11], in which some sufficient conditions were derived to ensure that the whole switched system is ISS when each subsystem is ISS. Based on piecewise Lyapunov-Krasovskii functional method, an explicit condition was provided to guarantee the ISS of the system under asynchronous switching in [12]. The references mentioned above focused on deterministic switched systems. Stochastic switched systems have recently been the focus of attention in control engineering such as [13], [14]. The stability of stochastic nonlinear systems with state-dependent switching was investigated in [15], where the switching laws can be viewed as a closed-loop control. Exponential stability and almost sure exponential stability of stochastic systems with state-dependent switching was

* This work is supported in part by the National Natural Science Foundation of China under Grant 61174058, National Science Fund for Distinguished Young Scholars of China under Grant 61325014, the Australian Research Council (DP140102180), the National Key Basic Research Program, China (2011CB710706, 2012CB215202), and the 111 Project (B12018).

studied in [16]. Sufficient conditions for stochastic versions of ISS for randomly switched systems without control inputs were provided in [17]. To the best of our knowledge, noise-to-state stability for random affine systems (studied in [8]) with switching has not been considered.

Inspired by [8], this paper is devoted to investigate the noise-to-state stability in probability for a class of random nonlinear systems with state-dependent switching. By adopting the similar method as in [15], a strong solution to the random switched system is constructed based on the corresponding state-dependent switching signal. Because of switching behavior between individual subsystems, the smoothness (about time) of Lyapunov function is destroyed, which means that the stability results to random systems without switching are not suitable for random switched systems. Based on a reasonable requirement for the random disturbance and some mild and easily verified conditions for every subsystem, the existence of global solution and the criteria on noise-to-state stability in probability to random switched systems are presented by applying Lyapunov function method.

The paper is organized as follows: Section II gives some preliminaries; In section III, the existence of global solution and the criteria on noise-to-state stability to random switched systems are developed by applying Lynapunov function technique; The paper is concluded in section IV.

Notions: The following notions are used throughout the paper. For a vector x, $|x|$ denotes the usual Euclidean norm; $\|X\|$ is the 2-norm of a matrix X; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; U_R denote the ball $|x| < R$ in $\mathbb{R}^n$. C^i denotes the set of all functions with continuous i-th partial derivative; $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes the set of all functions which are of class $\mathscr{K}$ and unbounded; $\mathscr{KL}$ denotes the set of all functions $\beta(s,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is of class $\mathscr{K}$ for each fixed t, and decreases to zero as $t \to \infty$ for each fixed s. For $a,b \in R$, denote $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. Function $\alpha : D \to \mathbb{R}$ is convex on D, if it satisfies

$$\frac{\alpha(s_1)+\alpha(s_2)}{2} \geq \alpha\left(\frac{s_1+s_2}{2}\right), \forall s_1,s_2 \in D.$$

II. Preliminary results

In this section, we will review some important results on the existence of solution and stability for random affine systems without switching.

Consider the following random affine system

$$\dot{x} = f(x(t),t) + g(x(t),t)\xi(t),\ x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, stochastic process $\xi(t) \in \mathbb{R}^l$ defined on the complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with a filtration $\mathscr{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets).

To guarantee the existence and uniqueness of solution for system (1), some assumptions are imposed on the stochastic process $\xi(t)$ and coefficients $f(x,t)$, $g(x,t)$ as in [8].

A1: Process $\xi(t)$ is $\mathscr{F}_t$-adapted and piecewise continuous, and satisfies

$$\sup_{t_0 \leq s \leq t} E|\xi(s)|^2 < \infty,\ \forall t \geq t_0. \tag{2}$$

H1: Both functions $f(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ are piecewise continuous in t, and locally Lipshitz in x, i.e., for any $R > 0$, there exists a constant $L_R \geq 0$ possibly dependent on R such that $\forall x_1, x_2 \in U_R, x_1 \neq x_2$,

$$|f(x_2,t) - f(x_1,t)| + \|g(x_2,t) - g(x_1,t)\| \leq L_R|x_2 - x_1|.$$

H2: There exists a constant $b_0 \geq 0$ such that

$$|f(0,t)| + \|g(0,t)\| < b_0.$$

For system (1), given any $k > 0$, define the first exit time from a region $U_k = \{x : |x| < k\}$ and its limit:

$$\eta_k = \inf\{t \geq t_0 : |x(t)| \geq k\},\ \eta_\infty = \lim_{k\to\infty} \eta_k, \tag{3}$$

with the special case $\inf\emptyset = \infty$.

The existence and uniqueness of the maximal solution are given by the following lemma.

Lemma 1 ([8]): Under assumptions **A1**, **H1** and **H2**, system (1) has a unique solution in the maximal existence interval $[t_0, \eta_\infty)$.

Based on some additional conditions, the following result can demonstrate that the maximal local solution is in fact a unique global one.

Lemma 2 ([8]): For system (1), under assumptions **A1**, **H1** and **H2**, if there exist a positive function $V(x(t),t) \in C$ and constants c,d such that for all $t \geq t_0$,

$$\lim_{k\to\infty} \inf_{|x|>k} V(x,t) = \infty, \tag{4}$$

and

$$E\left[V(x(t\wedge\eta_k), t\wedge\eta_k)\right] \leq de^{ct},\ \forall k > 0, \tag{5}$$

then system (1) has a unique solution $x(t)$ on $[t_0, \infty)$.

Definition 1: The stochastic process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is called strongly bounded in probability, if for any $\varepsilon > 0$, there exists an $r > 0$ such that

$$P\left\{\sup_{t\geq t_0} |x(t)| > r\right\} \leq \varepsilon.$$

A stricter condition was given to present the notions of noise-to-state stability for system (1) in [8].

A2: Process $\xi(t)$ is $\mathscr{F}_t$-adapted and piecewise continuous, and there exists parameters $c_0, d_0 > 0$ such that

$$E|\xi(t)|^2 < d_0 e^{c_0 t},\ \forall t \geq t_0. \tag{6}$$

Remark 1: It should be noted that **A2** $\Rightarrow$ **A1** can be easily verified.

Regarding $\xi(t)$ as random disturbance, the following notion of noise-to-state stability was given in [8].

Definition 2: System (1) is said to be noise-to-state stable in probability (NSS-P) if for any $\varepsilon > 0$, there exist a class-$\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathscr{K}_\infty$ function $\gamma(\cdot)$ such that for all $t \in [t_0, \infty)$ and $x_0 \in \mathbb{R}^n$,

$$P\left\{|x(t)| \le \beta(|x_0|, t-t_0) + \gamma\left(\sup_{t_0 \le s \le t} E|\xi(s)|^2\right)\right\} \ge 1-\varepsilon. \tag{7}$$

Lemma 3: ([18, Stochastic Barbalat lemma]) For the system (1), suppose that there exists a unique solution $x(t)$ in the interval $[t_0, \infty)$, which is strongly bounded in probability. If there exists a continuous nonnegative function $W(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that

$$E\left[\int_{t_0}^{\infty} W(x(t))dt\right] < \infty,$$

then

$$P\left\{\lim_{t\to\infty} W(x(t)) = 0\right\} = 1, \ \forall x_0 \in \mathbb{R}^n.$$

Furthermore, if W is continuous and positive definite, there holds

$$P\left\{\lim_{t\to\infty} |x(t)| = 0\right\} = 1, \ \forall x_0 \in \mathbb{R}^n.$$

III. Noise-to-state stability analysis to random switched systems

In this section, we deal with random affine systems with state-dependent switching, where the switching event is triggered by the state crossing some switching surfaces.

Consider a family of random affine systems described by

$$\Sigma_p : \dot{x} = f_p(x(t),t) + g_p(x(t),t)\xi(t), \ x(t_0) = x_0, \ p \in \mathscr{P}, \tag{8}$$

where $\mathscr{P} = \{1, 2, \cdots, N\}$, stochastic process $\xi(t)$ satisfies the assumption **A2**, and functions f_p, g_p ($p \in \mathscr{P}$) satisfy assumptions **H1** and **H2**.

Suppose that the p-th subsystem is active on region $\Psi_p (p = 1, 2, \cdots, N)$, where $\bigcup_{p=1}^{N} \Psi_p = \mathbb{R}^n$ and $\Psi_i \cap \Psi_j = \emptyset (i \ne j)$. Define a state-dependent switching signal as

$$\sigma(t) = p_j, \ x(t) \in \Psi_{p_j}, \ j = 0, 1, 2, \cdots, j^*, \tag{9}$$

where $p_j \in \mathscr{P}$ with $p_j \ne p_{j+1}$, j^* is the number of switches and $j^* \le \infty$. Without loss of generality, let $x_0 \in \Psi_{p_0}$. Then, the random switched system generated by system family (8) and switching signal (9) can be presented nominally as

$$\dot{x} = f_\sigma(x,t) + g_\sigma(x,t)\xi(t), \ x(t) \in \Psi_\sigma. \tag{10}$$

To develop our main results, we assume that there is no jump in the state at switching instants and there is no Zeno behavior, i.e., there is finite a number of switches on every bounded interval of time.

Although every subsystem satisfies globally Lipschitz condition, the switched systems may not still satisfy the locally Lipschitz condition, which is the reason why the strong solution does not exist. A well-known example is given below to illustrate this case.

Example: Consider the famous Tanaka equation

$$dx(t) = \mathrm{sign}(x)dB(t), \ x(0) = 0.1, \tag{11}$$

where $B(t)$ is 1-dimensional Brownian motion and

$$\mathrm{sign}(x) = \begin{cases} +1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Note that the active regions $\Psi_1 = \{x : x \ge 0\}$ and $\Psi_2 = \{x : x < 0\}$ satisfy $\Psi_1 \cup \Psi_2 = \mathbb{R}$ and $\Psi_1 \cap \Psi_2 = \emptyset$. As demonstrated in [19, P.73], function $\mathrm{sign}(x)$ dose not satisfy the local Lipschitz condition, so (11) has no strong solution in time interval $[0, \infty)$.

As a consequence, before studying the stability of random switched systems, we first consider the existence of solution to random switched system (10). By a recursive procedure as similar to [15], we can construct a strong solution to the random switched system (10) in the maximal existence interval.

For any $l > 0$, define the stopping times recursively as follows

$$\begin{aligned}
\tau_0 =& t_0, \\
\eta_{0,l} =& \inf\left\{t : t \ge t_0, \ \left|\Phi_{p_0}(t,t_0) * x_0\right| \ge l\right\}, \\
\eta_{0,\infty} =& \lim_{l\to\infty} \eta_{0,l}, \\
\tau_1 =& \inf\left\{\tau_0 \le t < \eta_{0,\infty}, \Phi_{p_0}(t,t_0) * x_0 \notin \Psi_{p_0}\right\}, \\
& \cdots \\
\eta_{j,l} =& \inf\left\{t : t \ge \tau_j, \ \left|\Phi_{p_j}(t,\tau_j) * \cdots * \Phi_{p_0}(\tau_1,\tau_0) * x_0\right| \ge l\right\}, \\
\eta_{j,\infty} =& \lim_{l\to\infty} \eta_{j,l}, \\
\tau_{j+1} =& \inf\left\{\tau_j \le t < \eta_{j,\infty}, \Phi_{p_j}(t,\tau_j) * \cdots * \Phi_{p_0}(\tau_1,\tau_0) \right. \\
& \left. * x_0 \notin \Psi_{p_j}\right\},
\end{aligned} \tag{12}$$

with the special case $\inf\emptyset = \infty$ in the definition of $\eta_{j,l}$ and $\inf\emptyset = \eta_{j,\infty}$ in the definition of τ_{j+1}, where $\Phi_p(t_2,t_1)$ denotes the flow of the p-th individual system from t_1 to t_2, $j = 0, 1, 2, \cdots, j^*$. And j^* is defined as

$$j^* = \begin{cases} j_0, & \text{if there exists a finite integer } j_0 \\ & \text{such that } \tau_{j_0+1} \ge \eta_{j_0,\infty}, \\ \infty, & \text{otherwise}, \end{cases} \tag{13}$$

which implies that $\tau_{j^*+1} = \eta_{j^*,\infty}$ by compared with definitions of τ_{j+1}.

The state-dependent switching signal (9) can also be transformed into a time-dependent switching signal

$$\sigma(t) = p_j, \ t \in [\tau_j, \tau_{j+1}), \ j = 0, 1, 2, \cdots, j^*, \tag{14}$$

where τ_j are switching times, $\tau_0 = t_0$, $p_j \in \mathscr{P}$ with $p_j \ne p_{j+1}$, and j^* is the number of switches.

From the recessive procedure, the solution of the random switched system (10) is described by

$$x(t)=\Phi_{p_j}(t,\tau_j)*\cdots*\Phi_{p_0}(\tau_1,\tau_0)*x_0,\\ \forall\tau_j\le t<\tau_{j+1},\ j=0,1,\cdots,j^*,$$

that is

$$\begin{aligned}x(t)=&x(t_0)+\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} f_\sigma(x(s),s)ds\\&+\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} g_\sigma(x(s),s)\xi(s)ds,\ \forall t\in[t_0,\infty).\end{aligned}\tag{15}$$

According to Lemma 1, in every interval $[\tau_j,\tau_{j+1})$ $(0\le j\le j^*)$ the active system has a unique solution. Thus, the solution (15) of random switched system (10) is unique, continuous and $\mathscr{F}_t$-adapted in the maximal existence interval $[t_0,\tau_{j^*+1})$.

The criteria on noise-to-state stability in probability (NSS-P) are given by the following result.

Theorem 1: For random switched system (10), assume that there exist a function $V\in C^1$ and class-$\mathscr{K}_\infty$ functions α_1,α_2,α and constant $d>0$ such that

$$\alpha_1(|x|)\le V(x)\le\alpha_2(|x|),\tag{16}$$

$$\frac{\partial V}{\partial x}f_p(x,t)+d\left|\frac{\partial V}{\partial x}g_p(x,t)\right|^2\le-\alpha(|x|),\ x(t)\in\Psi_p.\tag{17}$$

Then for every $x_0\in\mathbb{R}^n$, there exists a unique solution $x(t)=x(t;x_0,t_0)$ to the switched system (10) on $[t_0,\infty)$. If $\alpha\circ\alpha_2^{-1}(\cdot)$ and $\alpha_2\circ\alpha_1^{-1}(\cdot)$ are convex functions, the random switched system is NSS-P.

Proof: According to Lemma 1, based on the above recursive procedure, a strong solution (15) can be constructed for random switched system (10) in the maximal existence interval $[t_0,\tau_{j^*+1})$. And $\eta_{j^*,\infty}=\tau_{j^*+1}$.

For any $t\in[t_0,\infty)$, we have $x(s)\in\Psi_{p_j}$, $s\in[t\wedge\tau_j,t\wedge\tau_{j+1})$. By (17), the derivative of V along system (10) satisfies that

$$\begin{aligned}\dot V(x(s))=&\frac{\partial V}{\partial x}f_{p_j}(x,s)+\frac{\partial V}{\partial x}g_{p_j}(x,s)\xi(s)\\\le&\frac{\partial V}{\partial x}f_{p_j}(x,s)+d\left|\frac{\partial V}{\partial x}g_{p_j}(x,s)\right|^2+\frac{1}{4d}|\xi(s)|^2\\\le&-\alpha(|x|)+\frac{1}{4d}|\xi(s)|^2.\end{aligned}\tag{18}$$

From (15) and (18), we conclude

$$\begin{aligned}E\left[V(x(t\wedge\tau_{j^*+1}))\right]\le&V(x_0)-E\left[\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}}\alpha(|x(s)|)ds\right]\\&+\frac{1}{4d}E\left[\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}}|\xi(s)|^2ds\right].\end{aligned}\tag{19}$$

Then, we can obtain

$$E\left[V(x(t\wedge\tau_{j^*+1}))\right]\le V(x_0)+\frac{d_0}{4c_0d}e^{c_0t}\le(V(x_0)+\frac{d_0}{4c_0d})e^{c_0t}.$$

From Lemma 2, $\tau_{j^*+1}=\eta_{j^*,\infty}=\infty$ a.s. can be obtained, and then the first result holds. Let us complete the proof of the second one.

According to (16) and (19), we have

$$\begin{aligned}V(x(t))-V(x_0)\le&-\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}}\alpha(|x(s)|)ds\\&+\frac{1}{4d}\sum_{j=0}^{j^*}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}}|\xi(s)|^2ds\\\le&\int_{t_0}^{t}\left[-\alpha(|x(s)|)+\frac{1}{4d}|\xi(s)|^2\right]ds\\\le&\int_{t_0}^{t}\left[-\alpha\circ\alpha_2^{-1}(V(x(s)))+\frac{1}{4d}|\xi(s)|^2\right]ds.\end{aligned}\tag{20}$$

Note that $\alpha\circ\alpha_2^{-1}(\cdot)$ is a convex function. Taking expectations first on both sides of (20), then according to Fubini's theorem and Jensen's inequality, we can get

$$\begin{aligned}&E\left[V(x(t))\right]-V(x_0)\\\le&\int_{t_0}^{t}\left[-\alpha\circ\alpha_2^{-1}(E\left[V(x(s))\right])+\frac{1}{4d}E|\xi(s)|^2\right]ds.\end{aligned}$$

By defining $v(t)=E\left[V(x(t))\right]$, the above inequality leads to

$$v(t)\le v_0+\int_{t_0}^{t}\left[-\alpha\circ\alpha_2^{-1}(v(s))+\frac{1}{4d}E|\xi(s)|^2\right]ds,\ \forall t\in[t_0,\infty),\tag{21}$$

where $v(t)\ge0$ is a deterministic and continuous function in t. The following proof is similar to the Theorem 2.2 in [4]. Define the set

$$\mathscr{R}_{t_0}=\left\{v(t)\in\mathbb{R}\middle|v(t)\le\bar\alpha^{-1}\left(\frac{q}{4d}\sup_{t\ge t_0}E|\xi(t)|^2\right)\right\},$$

where $\bar\alpha\triangleq\alpha\circ\alpha_2^{-1}$ and $q\ge1$ is a constant. Then, define $B=[t_0,T)$ as the time interval before $v(t)$ enters $\mathscr{R}_{t_0}$ for the first time, where $T=\inf\left\{t\ge t_0:v(t)\in\mathscr{R}_{t_0}\right\}$ is a deterministic time. In view of the definition of $\mathscr{R}_{t_0}$, for $t\in B=[t_0,T)$, it holds that

$$v(t)\ge\bar\alpha^{-1}\left(\frac{q}{4d}\sup_{t\ge t_0}E|\xi(t)|^2\right)\ge\bar\alpha^{-1}\left(\frac{q}{4d}E|\xi(t)|^2\right),$$

which together with (21) implies

$$v(t)\le v_0-(1-\frac{1}{q})\int_{t_0}^{t}\bar\alpha(v(s))ds,\ \forall t\in[t_0,T).\tag{22}$$

It is obvious that for any $t\in[t_0,\infty)$

$$E\left[V(x(t\wedge T))\right]\le V(x_0)\le\alpha_2(|x_0|),$$

then, according to Chebyshev's inequality, we have

$$P\left\{V(x(t\wedge T))\ge\delta(\alpha_2(|x_0|))\right\}\le\frac{\alpha_2(|x_0|)}{\delta(\alpha_2(|x_0|))}\le\varepsilon,\tag{23}$$

where class-$\mathscr{K}_\infty$ function $\delta(\cdot)$ is chosen such that ε can be arbitrarily small. Then, by (16) and (23), it yields that

$$P\{|x(t\wedge T)| \leq \alpha_1{}^{-1}(\delta(\alpha_2(|x_0|)))\} \geq 1-\varepsilon,\ \forall t\in[t_0,\infty). \tag{24}$$

From the convexity of $\bar{\alpha}=\alpha\circ\alpha_2^{-1}$ and Fubini's theorem, by (22) and (16), we have

$$E\left[\int_{t_0}^{t\wedge T}\bar{\alpha}(\alpha_1(|x(s)|)ds\right] \leq \int_{t_0}^{t\wedge T}\bar{\alpha}(v(s))ds \leq \frac{q}{q-1}V(x_0)<\infty.$$

If $T=\infty$, it is obtained that

$$E\left[\int_{t_0}^{\infty}\bar{\alpha}(\alpha_1(|x(s)|)ds\right] \leq \frac{q}{q-1}V(x_0)<\infty.$$

From Lemma 3, we have

$$P\left\{\lim_{t\to\infty}x(t)=0\right\}=1,$$

which, together with (24), means that, for any $\varepsilon'>0$ there exists a class-$\mathscr{KL}$ function $\beta(\cdot,\cdot)$ such that

$$P\{|x(t)|\leq\beta(|x_0|,t-t_0)\}\geq 1-\varepsilon',\ \forall t\in[t_0,T). \tag{25}$$

Now let us pay attention to the interval $[T,\infty)$. Since function $\alpha_2\circ\alpha_1^{-1}$ is convex, by the definitions of $\mathscr{R}_{t_0}$ and T, and (16), it follows that

$$\begin{aligned} E[V(x(t))] &\leq \left(\alpha_2\circ\alpha_1^{-1}\right)(E[\alpha_1(|x(t)|)]) \\ &\leq \left(\alpha_2\circ\alpha_1^{-1}\right)(v(t)) \\ &\leq \left(\alpha_2\circ\alpha_1^{-1}\right)\circ\bar{\alpha}^{-1}\left(\frac{q}{4d}\sup_{t\geq t_0}E|\xi(t)|^2\right), \end{aligned} \tag{26}$$

for all $t\in[T,\infty)$. According to Chebyshev's inequality, the above inequality implies

$$P\left\{|x(t)|\leq\gamma\left(\sup_{t\geq t_0}E|\xi(t)|^2\right)\right\}\geq 1-\varepsilon'',\ \forall t\in[T,\infty), \tag{27}$$

where class-$\mathscr{K}$ function $\gamma(\cdot)$ is chosen such that ε'' can be made arbitrarily small. By causality, (27) together with (25), gives

$$P\left\{|x(t)|\leq\beta(|x_0|,t-t_0)+\gamma\left(\sup_{t_0\leq s\leq t}E|\xi(s)|^2\right)\right\}\geq 1-\varepsilon,$$

$\forall t\in[t_0,\infty)$, where $\varepsilon=\varepsilon'\wedge\varepsilon''$, which completes the proof. ■

IV. CONCLUSIONS

Noise-to-state stability for random affine systems with state-dependent switching has been investigated in this paper. Although the random switched system does not satisfy locally Lipschitz condition, a local maximal solution can be constructed by a recursive procedure under some mild and easily verified conditions. Based on a reasonable requirement to the stochastic disturbance, the existence of global solution and the criteria on noise-to-state stability in probability of random switched systems were presented by applying Lynapunov function method.

REFERENCES

[1] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.

[2] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.

[3] ——, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[4] M. Krstić and H. Deng, *Stability of Nonlinear Uncertain Systems*. New York: Springer, 1998.

[5] H. Deng, M. Krstić, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1237–1253, 2001.

[6] Z. J. Wu, X. J. Xie, and S. Y. Zhang, "Adaptive backstepping controller design using stochastic small-gain theorem," *Automatica*, vol. 43, no. 4, pp. 608–620, 2007.

[7] S. J. Liu, J. F. Zhang, and Z. P. Jiang, "A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems," *Acta Mathematicae Applicatae Sinica, English Series*, vol. 24, no. 1, pp. 141–156, 2008.

[8] Z. J. Wu, "Criteria on lyapunov stability of random affine systems and applications to controller design," *Submitted to IEEE Transactions on Automatic Control*.

[9] D. Liberzon, *Switching in Systems and Control*. Boston: MA: Birkhauser, 2003.

[10] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[11] W. X. Xie, C. Y. Wen, and Z. G. Li, "Input-to-state stabilization of switched nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1111–1116, 2001.

[12] Y. E. Wang, X. M. Sun, P. Shi, and J. Zhao, "Input-to-state stability of switched nonlinear systems with time delays under asynchronous switching," *IEEE Transactions on Cybernetics*, vol. 43, no. 6, pp. 2261–2265, Dec 2013.

[13] M. Margaliot, "Stability analysis of switched systems using variational principles: An introduction," *Automatica*, vol. 42, no. 12, pp. 2059–2077, 2006.

[14] W. Feng, J. Tian, and P. Zhao, "Stability analysis of switched stochastic systems," *Automatica*, vol. 47, no. 1, pp. 148–157, 2011.

[15] Z. J. Wu, M. Y. Cui, P. Shi, and H. Karimi, "Stability of stochastic nonlinear systems with state-dependent switching," *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 1904–1918, 2013.

[16] H. Zhang, Z. J. Wu, and Y. Q. Xia, "Exponential stability of stochastic systems with hysteresis switching," *Automatica*, vol. 50, no. 2, pp. 599–606, 2014.

[17] D. Chatterjee and D. Liberzon, "Towards iss disturbance attenuation for randomly switched systems," in *46th IEEE Conference on Decision and Control*, Dec 2007, pp. 5612–5617.

[18] Z. J. Wu, Y. Q. Xia, and X. J. Xie, "Stochastic barbalat's lemma and its applications," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1537–1543, 2012.

[19] B. Øksendal, *Stochastic differential equations*. Springer, 2003.